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THE SOLUTION PROCEDURE FOR THE ORANI MODEL EXPLAINED BY A SIMPLE EXAMPLE

by

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Contents

Tables

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1. Introduction

ORANI is a large multisectoral model of the Australian economy.[1] In standard applications it identifies 113 industries, 230 commodities (115 domestically produced and 115 imported), 9 types of labour, 7 types of agricultural land and 113 types of capital (one for each industry). It contains explicit modelling, at this disaggregated level, of many types of commodity and factor flows, e.g., inputs to current production, inputs to capital creation, household consumption, exports and margin services (retail, wholesale and transport). The multi-product characteristics of production in Australian agricultural industries are also explicitly modelled, and a facility is included for disaggregating economy-wide results to the regional (State) level.

The reason for including so much detail in ORANI is to facilitate its use by a variety of Government agencies with interests in different spheres of economic policy. Among the agencies which have used the model are the Industries Assistance Commission, the

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1. ORANI is fully documented in Dixon, Parmenter, Sutton and Vincent (1982), cited hereafter as DPSV (1982).

Bureau of Agricultural Economics, the Bureau of Industry Economics and the Premier's Department of South Australia. Applications of ORANI made by these agencies and other groups include simulations of the effects on industries, occupations and regions of changes in tariffs, the exploitation of mineral resources, changes in world commodity prices, changes in the exchange rate, the adoption of import parity pricing for oil products, subsidies to ailing industries, the move towards equal pay for women, changes in real wages and the adoption of Keynesian demand stimulation policies. Each of these applications draws upon different aspects of the model's detail.

To solve such a large model, we followed initially the method pioneered by Johansen (1960). This method which relies on linear approximations enables us to obtain solutions at modest cost. It has the additional advantages of allowing flexibility in the selection of exogenous variables and of facilitating changes to the structure of the model. Recently we have extended the Johansen method in a way which retains its principal advantages but which allows elimination of errors introduced by linear approximation.

In section 2 of this paper we describe our extended Johansen method by means of an elementary example. Complete descriptions of the mathematical theory[1] underlying the method and the application of the method to the ORANI model are in DPSV (1982, ch.5).

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1. The relevant mathematics is that of Euler's method for numerical solution of systems of differential equations supplemented by Richardson's extrapolation.

Our aim in this paper is to provide a quick introduction to the method for readers whose time constraints and interests would make more detailed reading impractical. Section 3 contains concluding remarks.

2. The extended Johansen method for computing solutions for a general equilibrium model : an illustrative example

The ORANI model can be thought of as a system of m equations in n variables of the form

$$F(V) = 0 , \qquad (1)$$

where F is a vector function of length m and V is a vector of length n. The number of variables n exceeds the number of equations m.

System (1) imposes conditions such as : demands equal supplies for goods and factors of production, prices reflect costs, demands reflect prices and incomes, and supplies reflect profit maximizing decisions. For the purposes of illustrating the computational approach adopted in ORANI it is not necessary to describe further the economics underlying system (1). Sufficient motivation can be provided by a simple example devoid of economic content. Let us assume that system (1) consists of 2 equations in 3 variables and has the form :

$$V_1^2 V_3 - 1 = 0 , \qquad (2.1)$$

$$V_1 + V_2 - 2 = 0 . \qquad (2.2)$$

Because system (1) contains more variables than equations we assign exogenously-given values to $(n-m)$ variables and solve for the remaining m, the endogenous variables. In applications of ORANI, many different allocations of the variables between the exogenous and endogenous categories have been made. For example, if we are using ORANI in an analysis of the effects of changing

References

Dixon, P.B., Parmenter, B.R., Sutton, J., and Vincent, D.P. (1982), ORANI, A Multisectoral Model of the Australian Economy, North Holland Publishing Company. Cited in text as DPSV (1982).

Johansen, L. (1960), A Multi-Sectoral Study of Economic Growth, North Holland Publishing Company, Amsterdam (2nd edition 1974).

evaluation of the $A(V)$ matrix (the matrix on the LHS of (8) in our example of section 2), one extra computation of the elasticities matrix $B(V)$ and one application of a simple extrapolation procedure such as (17).[1]

Compared with various non-linear procedures often used in solving general equilibrium models, the Johansen method has several important advantages.[2] First, because only simple matrix operations are required, model size presents no computational problem. Data limitations constrain the size of system (5) long before computational considerations play a role. Second, maximum flexibility is retained by the user of the model in his choice of exogenous and endogenous variables. Changes in this choice are easily handled by a re-partitioning of the $A(V^I)$ matrix, see equation (10). Finally model development can be accommodated without requiring the re-thinking of algorithms or extensive reprogramming. Often all that is required is a re-dimensioning of the $A(V)$ matrix. The advantages of the Johansen method are retained by our extended Johansen method.

1. Computational costs are discussed in DPSV (1982, section 34).
2. Alternatives to the Johansen method are discussed in DPSV (1982, section 8).

the tariff on footwear, then this variable is exogenous. On the other hand, if we are using ORANI to calculate the change in the tariff which would be required to ensure a given level of footwear employment, then the footwear tariff is an endogenous variable and footwear employment is exogenous. For our illustrative system (2.1) - (2.2), we will assume in this paper that the exogenous variable is V_3 and the endogenous variables are V_1 and V_2.

With this assignment of the variables to the exogenous and endogenous categories, we can easily derive solution equations for the system (2.1) - (2.2). That is, we can express the endogenous variables as functions of the exogenous variable as follows :

$$V_1 = V_3^{-\frac{1}{2}} \qquad (3.1)$$

and

$$V_2 = 2 - V_3^{-\frac{1}{2}} , \qquad (3.2)$$

where we assume (as is often the case in economic models) that only positive values for the variables are of interest.[1] With a solution system such as (3.1) - (3.2), we have no difficulty in evaluating the effects on the endogenous variables of shifts in the exogenous variable. For example, assume that we are initially in a situation where

$$V^I \equiv (V_1^I, V_2^I, V_3^I) = (1,1,1) . \qquad (4)$$

Notice that V^I satisfies (2.1) - (2.2). Then we want to evaluate the effects on V_1 and V_2 (employment and prices, say) of a shift in V_3 (the level of protection) from 1 to 1.1. By substituting

1. Thus in deriving (3.1) from (2.1) we can ignore the possibility of negative or complex roots.

into (3.1) and (3.2), we find that the new values for V_1 and V_2 are .9535 and 1.0465. We conclude that a 10 per cent increase in V_3 induces a 4.65 per cent reduction in V_1 and a 4.65 per cent increase in V_2.

The ORANI computations are similar to that just described in one respect. They make use of an initial solution such as (4) with results being reported as percentage deviations from this initial solution. The initial solution (i.e. the initial values for prices, quantities, tariffs, etc.) is known from the data used in setting the parameters of system (1).[1] In another respect, the ORANI computations differ from the simple approach using (3.1) and (3.2). This is because the complexity and size of the ORANI system (1) rule out the possibility of deriving explicit solution equations. In other words, in the context of ORANI we cannot make the step from (2.1) - (2.2) to (3.1) - (3.2).

Instead, we follow the approach of Johansen (1960) and linearize system (1). That is, we derive from (1) a system of the form:

$$A(V^I)v = 0 , \tag{5}$$

where $A(V^I)$ is a $m \times n$ matrix whose components are functions of V evaluated at V^I, the initial values of the variables. The $n \times 1$ vector v shows the percentage changes in the variables V.

The derivation of (5) can be illustrated by returning to (2.1) - (2.2). We totally differentiate the LHS's of (2.1) and

1. See DPSV (1982, pp. 201-202).

Table 2. Industry-Output Effects of a 100 per cent Across-the-Board Tariff Cut Computed in the ORANI Model via the Extended Johansen Method (main gaining and losing industries only)

Industry identi-fication number	(1) Computed Projections(a) 1-step	(2) 2-step	(3) 4-step	(4) 8-step	(5) 16-step	(6) Extrapolations(b) 1-2 steps	(7) 8-16 steps	(8) Percentage Errors Johansen $[(\frac{(1)}{(7)}-1]100$	(9) 1-2 step extrap'n. $[(\frac{(6)}{(7)}-1]100$
14	14.41633	14.78293	14.96247	15.05172	15.09640	15.14953	15.14108	-4.79	0.06
25	13.34001	13.98722	14.34011	14.52664	14.62299	14.63443	14.71933	-9.37	-0.58
11	10.02261	10.10874	10.16028	10.19008	10.20635	10.19487	10.22263	-1.96	-0.27
64	8.88825	9.08601	9.19812	9.25948	9.29187	9.28377	9.32426	-4.68	-0.43
63	8.59218	9.38494	9.82725	10.06176	10.18273	10.17769	10.30370	-16.61	-1.22
13	7.30533	7.37633	7.42870	7.46112	7.47926	7.44733	7.49741	-2.56	-0.67
76	7.28429	6.70584	6.43846	6.31306	6.25293	6.12740	6.19280	17.63	-1.06
6	5.61351	5.87919	6.02332	6.09933	6.13856	6.14487	6.17780	-9.13	-0.53
3	5.56409	5.65745	5.71728	5.75197	5.77079	5.75081	5.78960	-3.90	-0.67
18	4.92539	5.03666	5.10865	5.15050	5.17321	5.14794	5.19592	-5.21	-0.92
41	-3.90072	-4.13497	-4.26576	-4.33476	-4.37051	-4.36921	-4.40553	-11.46	-0.82
50	-4.03336	-4.28160	-4.40859	-4.47327	-4.50596	-4.52984	-4.53865	-11.13	-0.19
67	-6.18812	-6.75976	-7.08976	-7.26824	-7.36118	-7.33141	-7.45412	-16.98	-1.65
28	-7.07837	-7.68314	-8.01120	-8.18385	-8.27268	-8.28791	-8.36152	-15.35	-0.88
79	-7.24562	-8.47378	-9.36506	-9.91805	-10.22755	-9.70193	-10.53704	-31.24	-7.93
31	-9.25475	-9.92945	-10.27842	-10.45820	-10.54992	-10.60415	-10.64163	-13.03	-0.35
73	-9.63989	-10.25516	-10.61353	-10.80751	-10.90845	-10.87043	-11.00939	-12.44	-1.26
68	-10.37971	-11.15557	-11.54041	-11.73343	-11.83038	-11.93143	-11.92733	-12.98	0.03
39	-10.52144	-13.35426	-15.54377	-16.94308	-17.73697	-16.18709	-18.53086	-43.22	-12.65
32	-12.68090	-13.49826	-13.91897	-14.13490	-14.24474	-14.31561	-14.35459	-11.66	-0.77

(a) All projections are percentage changes from the initial values of the variables. Column (1) contains results from a Johansen-style solution. Columns (2) - (5) were computed via the n-step extended Johansen method for n = 2, 4, 8 and 16 respectively.

(b) Column (6) was calculated from columns (1) and (2) via extrapolations similar to those described in equations (17) and (18). Column (7) was calculated in an analogous way from columns (4) and (5). The results in column (7) are assumed to be free from linearization errors and are used as the exact solution to the model in calculating the percentage errors given in columns (8) and (9).

3. Conclusion

Table 2 is reproduced from DPSV (1982, p. 332). It shows computations from the ORANI model of the effects on industry outputs of an elimination of tariffs. The computations have been performed with a 1-step procedure (the Johansen method) and with 2, 4, 8 and 16 step procedures (the extended Johansen method). Extrapolations similar to those described in equations (17) and (18) are shown in columns (6) and (7). We accept the 8-16 extrapolation as being the correct solution (i.e., the free from linearization error).[1] Column (8) shows the percentages errors in the Johansen-style computation. Column (9) shows that these errors are sharply reduced by a 1-2 step extrapolation.

The policy shock being simulated in table 2 is a large one, much larger than those routinely examined in applications of the ORANI model. Consequently, the Johansen-style errors shown in column (8) of table 2 are larger than those normally encountered. Our experience with the model has been that Johansen-type computations are usually adequate. Certainly we have found that 1-2 step extrapolations reduce linearization errors well beyond the point where they could have any practical significance. Thus, our extended Johansen method enables us to eliminate uncertainties concerning linearization errors for the computational cost of one extra

1. That this is a highly accurate solution has been confirmed by substitution back into the ORANI system (1).

(2.2). Then we set these total differentials to zero recognizing that if (2.1) and (2.2) are to continue to be satisfied after a disturbance in the exogenous variables, then the changes in their LHS's must be zero. Thus, we write

$$2V_1 V_3(dV_1) + V_1^2(dV_3) = 0 \qquad (6.1)$$

and

$$dV_1 + dV_2 = 0 \ , \qquad (6.2)$$

or equivalently, in linear-percentage-change form, we write

$$2v_1 + v_3 = 0 \qquad (7.1)$$ [1]

and

$$\left(\frac{V_1}{2}\right)v_1 \quad \left(\frac{V_2}{2}\right)v_2 = 0 \ , \qquad (7.2)$$ [2]

where $v_i = 100(dV_i/V_i)$, $i=1,2,3$. In matrix notation, (7.1) and (7.2) are

$$\begin{pmatrix} 2 & 0 & 1 \\ & & \\ V_1/2 & V_2/2 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = 0 \ . \qquad (8)$$

With V set at its initial value, V^I given in (4), equation (8) becomes

$$\begin{pmatrix} 2 & 0 & 1 \\ & & \\ .5 & .5 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = 0 \ . \qquad (9)$$

1. (7.1) is derived from (6.1) by dividing through by $V_1^2 V_3$. We assume that $V_i > 0$ for all i.

2. We divide through by 2. This is not necessary. It is customary, however, to use share coefficients in the linear-percentage-change system. Notice that $V_1/2$ and $V_2/2$ are the shares of V_1 and V_2 in the sum of V_1 and V_2.

This is a system of the form (5) for the model (2.1) - (2.2) with the initial solution (4).

In a Johansen-style computation, system (5) effectively replaces system (1) as the model. In evaluations of how far the endogenous variables will move from their initial values in response to given movements in the exogenous variables, system (5) is rewritten as:

$$A_\alpha(V^I)v_\alpha + A_\beta(V^I)v_\beta = 0 , \qquad (10)$$

where v_α is the $m \times 1$ vector of percentage changes in the endogenous variables, v_β is the $(n - m)$ vector of percentage changes in the exogenous variables and $A_\alpha(V^I)$ and $A_\beta(V^I)$ are appropriate submatrices of $A(V^I)$, i.e. $A_\alpha(V^I)$ is the $m \times m$ matrix formed by the columns of $A(V^I)$ corresponding to the endogenous variables and $A_\beta(V^I)$ is the $m \times (n - m)$ matrix formed by the columns corresponding to the exogenous variables. Then (10) is solved for v_α in terms of v_β by matrix inversion,[1] giving

$$v_\alpha = -A_\alpha^{-1}(V^I)\, A_\beta(V^I)v_\beta \qquad (11)$$

or more compactly

$$v_\alpha = B(V^I)v_\beta . \qquad (12)$$

The typical element, $B_{ij}(V^I)$, of $B(V^I)$ is the elasticity in the region of V^I of the i^{th} endogenous variable with respect to changes in the j^{th} exogenous variable. That is, $B_{ij}(V^I)$ can

1. We assume that the relevant inverse exists. If this is not true, then the Johansen method will fail. However, it is likely that if $A_\alpha(V^I)$ is singular, then our classification of endogenous and exogenous variables is illegitimate. That is, it is likely that system (1) cannot be solved for V_α in terms of V_β in the region of V^I. See DPSV (1982, section 35).

Table 1. Various Solutions for V_1 and V_2 in the System (2.1) - (2.2) when V_3 is Moved from 1 to 2

Endogenous variables	Initial values	Estimated values after an increase in V_3 from 1 to 2					
		1 step computation	2 step computation	4 step computation	1, 2 step extrapolation(a)	2, 4 step extrapolation(b)	Truth(c)
V_1	1	.5	.625	.6703	.75	.716	.707
V_2	1	1.5	1.375	1.3297	1.25	1.284	1.293

(a) Computed according to (17).

(b) Computed according to (18).

(c) Computed using (3.1) and (3.2).

Fortunately, however, effective extrapolation techniques can be applied to give a high degree of accuracy even when our extended Johansen method is applied with only a small number of steps. Consider for example table 1 where we have set out the results of our various computations of the effects on V_1 and V_2 in the system (2.1) - (2.2) of an increase in V_3 from 1 to 2. Notice that when we double the number of steps in our extended Johansen computations, we approximately halve our errors. With a 1 step computation, our errors are -.207 for V_1 (i.e. .5 - .707) and .207 for V_2 (i.e. 1.5 - 1.293). With a 2 step computation our errors are -.082 and .082. With 4 steps the errors are -.037 and .037. This suggests that we can estimate V_1 and V_2 by:

$$V_i(1,2) = V_i(2) + (V_i(2) - V_i(1)), \quad i=1,2 \tag{17}$$

or

$$V_i(2,4) = V_i(4) + (V_i(4) - V_i(2)), \quad i=1,2 \tag{18}$$

where $V_i(n)$ is the estimate for V_i from an n step computation and $V_i(n,2n)$ is an improved estimate based on extrapolation using results from n step and $2n$ step computations. The values for $V_i(1,2)$ and $V_i(2,4)$ are set out in table 1 under the headings "1,2 step extrapolation" and "2,4 step extrapolation". It is clear that the extrapolations have sharply reduced the errors associated with the extended Johansen method.

be interpreted as the percentage change in $(V_\alpha)_i$ which would result from a one per cent increase in $(V_\beta)_j$ where the initial values for all variables are given by V^I.

The computations (10) - (12) can be illustrated via system (9). Where variable 3 is exogenous, we rewrite (9) as

$$\begin{pmatrix} 2 & 0 \\ .5 & .5 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} v_3 = 0 \ . \tag{13}$$

Then,

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = - \begin{pmatrix} 2 & 0 \\ .5 & .5 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} v_3 \ . \tag{14}$$

That is

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} -.5 \\ .5 \end{pmatrix} v_3 \ . \tag{15}$$

Equation (15) indicates that in the region of $V^I = (1,1,1)$ the elasticity of variable 1 with respect to variable 3 is -.5 and the elasticity of variable 2 with respect to variable 3 is .5. By using (15) we would say that a 10 per cent increase in V_3 would induce a 5 per cent reduction in V_1 and a 5 per cent increase in V_2. This is close to the answers (-4.65 and 4.65) which we found earlier by substituting into (3.1) and (3.2).

A little experimentation with (15) indicates that the Johansen approach is satisfactory for computing the effects on the endogenous variables of small changes in the exogenous variables. However, when we make large changes in V_3, (15) may not give a satisfactory approximation to the effects on V_1 and V_2. For example, assume that we increase V_3 by 100 per cent (i.e. from

1 to 2). Then (15) implies that V_1 will fall by 50 per cent and V_2 will increase by 50 per cent. The correct values derived from (3.1) and (3.2) are that V_1 will fall by 29.3 per cent while V_2 will increase by 29.3 per cent.

The weakness of the Johansen method is that it fixes the elasticities of the endogenous variables with respect to the exogenous variables at their initial values, $B_{ij}(V^I)$. As we move away from V^I we should, ideally, allow the elasticities to move. Thus, when faced with a large change in the exogenous variables, our approach in ORANI is to make a sequence of Johansen-style computations. For example, if we want to evaluate the effects of a 100 per cent tariff increase, we can first use (12) to generate the effects of a 50 per cent increase. This would take us from the initial situation $(V = V^I)$ to one where $V = V^I + \Delta V_{50}$ with ΔV_{50} denoting our estimate of the change in V arising from the first half of the tariff increase. Then we can re-evaluate the elasticity matrix B at $V = V^I + \Delta V_{50}$ and use the re-evaluated matrix in computing the effects of the second half of the tariff increase.

To illustrate our extended Johansen method, we reconsider the problem of calculating the effects on V_1 and V_2 of an increase in V_3 from 1 to 2 in the model (2.1) - (2.2) with the starting condition (4). We tackle the problem in two steps. First we increase V_3 by 50 per cent from 1 to 1.5. According to (15), this will reduce V_1 by 25 per cent (i.e. from 1 to .75) and increase V_2 by 25 per cent (i.e. from 1 to 1.25). Next we revaluate the

B matrix at $(V_1, V_2, V_3) = (.75,\ 1.25,\ 1.50)$. This requires re-evaluating the $A(V)$ matrix on the LHS of (8). We obtain:

$$A(.75,\ 1.25,\ 1.50) = \begin{pmatrix} 2 & 0 & 1 \\ .375 & .625 & 0 \end{pmatrix}$$

which leads to

$$B(.75,\ 1.25,\ 1.50) = \begin{pmatrix} 2 & 0 \\ .375 & .635 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -.5 \\ .3 \end{pmatrix} . \quad (16)$$

In the second step of the computation we use (16) in calculating the effects of moving V_3 from 1.5 to 2. For this 33 per cent increase in V_3, the responses indicated by equation (16) are a 16.66 per cent reduction in V_1 (i.e. from .75 to .625) and a 10 per cent increase in V_2 (i.e. from 1.25 to 1.375). Thus, our conclusion from the two-step computation is that a 100 per cent increase in V_3 (from 1 to 2) induces a 37.5 per cent decrease in V_1 (from 1 to .625) and a 37.5 per cent increase in V_2 (from 1 to 1.375).

Although our 2 step procedure has taken us closer to the correct answer than the one-step Johansen computation, we still have uncomfortably large errors. (Recall that the correct answer is that V_1 decreases by 29.3 per cent and V_2 increases by 29.3 per cent). On carrying out a 4 step computation (i.e. by increasing V_3 from 1 to 1.25, then from 1.25 to 1.5, etc.) we found that our extended Johansen method put the decrease in V_1 at 32.97 per cent (i.e. from 1 to .6703) and the increase in V_3 at 32.97 per cent (i.e. from 1 to 1.3297). Further gains in accuracy can be achieved by further increasing the number of steps in our computations.