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# The Mathematical Programming Approach to Applied General Equilibrium Modelling: Notes and Problems

by

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## *Preface*

The mathematical programming approach to applied general equilibrium analysis, although no longer the dominant tool, is still useful, from at least two points of view:

- it neatly integrates into an economy-wide framework the microeconomic theory of the behaviour of agents constrained by inequalities; and
- it provides a useful approach for computing the solutions of some general equilibrium problems not solvable with the current *GEMPACK* software (see, e.g., Dixon (1991), cited below on p. 16).

The material contained in this paper was meant to be included in our forthcoming graduate-level text (Peter B. DIXON, B.R. PARMENTER, Alan A. POWELL and P.J. WILCOXEN, *Notes and Problems in Applied General Equilibrium Economics*, Amsterdam, North-Holland), but space limitations led to our reluctant exclusion of it from the text. Publication in the Impact series will mean that those who find the approach in our textbook useful will be able to apply the same method towards mastering mathematical programming in a general equilibrium context.

Peter B. Dixon
April 1991

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# *The Mathematical Programming Approach to Applied General Equilibrium Modelling: Notes and Problems*

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## *1 Introduction*

General equilibrium models can often be formulated as mathematical programming (i.e., constrained optimization) problems. To illustrate this approach, we consider a 2-consumer, 2-good, pure trade (no production) model in which the initial endowments are

$$Z_1 = \begin{pmatrix} 100 \\ 0 \end{pmatrix} \quad , \qquad Z_2 = \begin{pmatrix} 0 \\ 100 \end{pmatrix} \quad , \qquad (1.1)$$

i.e., consumer 1 owns 100 units of good 1 and consumer 2 owns 100 units of good 2. We assume that consumer 1's preferences are described by the utility function

$$U_1(C_1) = ln(C_{11}) + ln(C_{12}) \qquad (1.2)$$

where $C_{11}$ and $C_{12}$ are his consumption levels for goods 1 and 2 and $C_1$ is the vector $(C_{11}, C_{12})'$. Similarly, we assume that consumer 2's utility function is

$$U_2(C_2) = ln(C_{21}) + ln(C_{22}) \ . \qquad (1.3)$$

The problem of solving this model is to find non-negative values for the product prices (denoted by the vector $P' \equiv (P_1, P_2)$), the consumption vectors ($C_1$ and $C_2$) and the consumer incomes ($Y_1$ and $Y_2$) which jointly satisfy the conditions:

$$C_i \text{ maximizes } U_i(C_i) \text{ subject to } P'C_i \le Y_i \ , \qquad i=1,2 \ , \ (i)$$

$$\sum_{i=1}^{2} C_i - \sum_{i=1}^{2} Z_i \le 0 \ , \qquad (ii)^1$$

1 Notice that the right hand side of condition (ii) is a 2 ×1 vector of zeros. The right hand side of (iii) is a scalar. Although we use the same symbol, namely 0, the distinction is clear from the context.

$$P'\left(\sum_{i=1}^{2} C_i - \sum_{i=1}^{2} Z_i\right) = 0 \quad , \qquad \text{(iii)}$$

and

$$P'Z_i = Y_i \quad , \quad i=1,2 \quad . \qquad \text{(iv)}$$

Condition (i) says that consumers maximize their utilities subject to their budget constraints. Condition (ii) says that demand for each good is less than or equal to supply. This condition combined with (iii) implies that goods in excess supply have zero price. Condition (iv) says that consumers' incomes are the values of their initial endowments. A price normalization condition must be added if we want to tie down the absolute values for $P_1$ and $P_2$. We also know from Walras' law that either the market clearing condition for one of the goods or the definition of income for one of the consumers could be eliminated. For the present, however, we will leave the model in the form (i) – (iv).

Perhaps the most obvious approach to solving general equilibrium models is via excess demand functions. Applying this approach to the model (i) - (iv), we first derive the consumer demand equations from condition (i). Under (1.2) and (1.3) we obtain[2]

$$C_{ij} = \frac{1}{2}\,\frac{Y_i}{P_j} \quad , \quad i,j=1,2 \quad . \qquad (1.4)$$

Next we use (iv) to eliminate the $Y_i$'s, i.e., we write (1.4) as

$$C_{ij} = \frac{1}{2}\,\frac{P'Z_i}{P_j} \quad , \quad i,j=1,2 \quad . \qquad (1.5)$$

With the $Z_i$s given by (1.1), (1.5) reduces to

$$\left.\begin{array}{l} C_{1j}=50P_1/P_j, j=1,2, \\ C_{2j}=50P_2/P_j, j=1,2. \end{array}\right\} \qquad (1.6)$$

---

2 If $P_j$ were zero, then under (1.2) and (1.3), demand for good j would be unlimited. Because supply is finite, we may assume (in view of condition (ii)) that neither price is zero.

On substituting from (1.6) and (1.1) into (ii) we obtain the system of excess demand equations[3]

$$50\ \frac{P_1}{P_1} + 50\ \frac{P_2}{P_1} - 100 = 0\ , \tag{1.7}$$

$$50\ \frac{P_1}{P_2} + 50\ \frac{P_2}{P_2} - 100 = 0\ , \tag{1.8}$$

At this stage, we introduce a normalization rule, e.g.,

$$P_1 = 1\ . \tag{1.9}$$

With this particular rule, (1.7) and (1.8) imply that

$$P_2 = 1\ .$$

Substituting back into (iv) gives $Y_1 = Y_2 = 100$. We complete the solution by substituting into (1.6), obtaining $C_{ij} = 50$ for all i,j.

However, rather than using excess demand functions, here we will deduce solutions to general equilibrium models via mathematical programming problems. In our illustrative model, (i) - (iv), we can use the problem of choosing non-negative values for $C_{ij}$, i,j=1,2

to maximize

$$\sum_{i=1}^{2} w_i\ U_i\ (C_i) \tag{1.10}$$

subject to

$$\sum_{i=1}^{2} C_i - \sum_{i=1}^{2} Z_i \leq 0\ , \tag{1.11}$$

3 Because neither price is zero, we may assume that condition (ii) is satisfied by equalities.

where the $w_i$s are weights satisfying

$$\sum_{i=1}^{2} w_i = 1 \quad . \tag{1.12}$$

In problem (1.10) – (1.11), we allocate the available commodities to maximize a weighted average of the utilities of the two consumers. We try to set the weights so that the consumption vectors allocated to the consumers are consistent with their budget constraints. In the present example where (1.1), (1.2) and (1.3) apply, it is intuitively clear that we should set $w_1 = w_2 = \frac{1}{2}$. Then (1.10) – (1.11) can be written as:

choose $C_{ij}$ , $i,j=1,2$

to maximize

$$\frac{1}{2}\left\{ln(C_{11}) + ln(C_{12})\right\} + \frac{1}{2}\left\{ln(C_{21}) + ln(C_{22})\right\} \tag{1.13}$$

subject to

$$\sum_{i=1}^{2} C_{ij} = 100 \ , \quad j=1,2 \quad . \qquad (1.14)^4$$

To solve (1.13) – (1.14) we note the if $\bar{C}_{ij}$, i,j=1,2, is a solution, then there exist $\bar{P}_1$, $\bar{P}_2$ $\geq 0$ such that

$$1/(2\bar{C}_{ij}) = \bar{P}_j \ , \quad i=1,2, \ j=1,2 \tag{1.15}$$

and

4 We have written the constraints as equalities. The maximization of (1.13) requires that all of the available supplies be used.

$$\sum_{i=1}^{2} \bar{C}_{ij} = 100 \quad , \quad j=1,2 \quad . \tag{1.16}$$

(1.15) – (1.16) imply that

$$\bar{C}_{ij} = 50 \text{ for } i=1,2, \; j=1,2 \tag{1.17}$$

and

$$\bar{P}_j = 0.01 \text{ for } j=1,2 \quad . \tag{1.18}$$

Thus, the solution to the problem (1.13) – (1.14), together with the associated Lagrangian multipliers, has revealed the solution to the model (i) – (iv).[5]

Why can we solve the model in this way? The main ingredient in the underlying theory is the utility possibilities set (Samuelson (1950)). The utility possibilities set shows all the combinations, ($U_1$, $U_2$), of utility for the two consumers which are possible given their combined resources. Figure1.1 illustrates the utility possibilities set in our example where the utility functions are (1.2) and (1.3) and the resource endowments are given in (1.1). Because there are no market imperfections, externalities or taxes in our model (i) – (iv), we know that solutions will be Pareto optimal. In terms of Figure1.1, the solution to (i) – (iv) will imply utility combinations on the frontier AA. Consequently, one way to solve the model is to search this frontier.

There are various methods for obtaining points on the utility possibilities frontier. Problem (1.10) – (1.11), for example, will indicate points on AA. At solutions to (1.10) – (1.11) the utility levels will be at points where the slope of AA is equal to the negative of the ratio of the weights $(-w_1/w_2)$ . (With $w_1 = w_2 = \frac{1}{2}$, the solution to (1.10) – (1.11) implies utility levels at B in Figure1.1.) We could

5 The absolute values (but not the relative values) obtained for the prices differ from those obtained when we worked via the excess demand functions. This is of no importance since our model (i) – (iv) does not determine absolute prices.

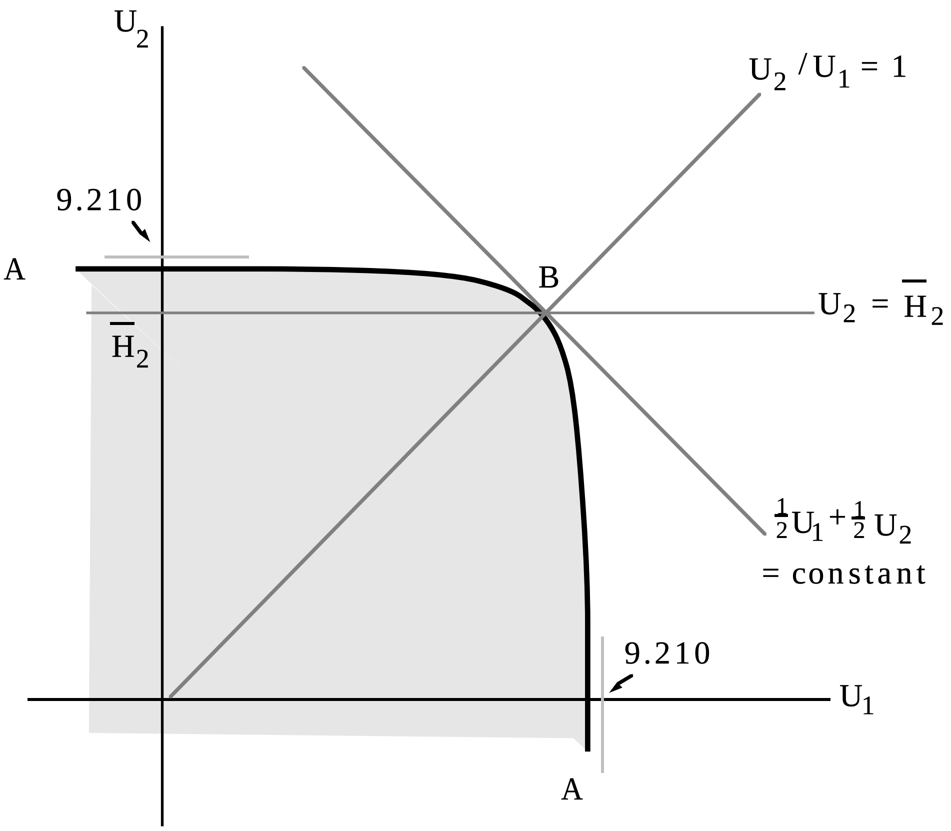



Figure1.1 Utility possibilities set

The utility functions are (1.2) and (1.3) and the combined resources of the two consumers are 100 units of each good, i.e., $Z_1 + Z_2 = (100, 100)$. Points on the frontier, AA, are Pareto optimal. At these points, it is impossible to increase the utility of one consumer without reducing the utility of the other. In our example, the equation for the utility possibilities frontier is

$$U_2 = 2\, ln \{100 - exp(U_1 / 2)\}.$$

You are asked to derive this equation in Exercise 3(d).

also generate points on AA by solving problems of the form: choose non-negative values for $C_{ij}$, i,j=1,2

to maximize

$$U_1(C_{11}, C_{12}) \tag{1.19}$$

subject to

$$\sum_{i=1}^{2} C_{ij} - \sum_{i=1}^{2} Z_{ij} \leq 0 \;, \quad j=1,2 \tag{1.20}$$

and

$$U_2(C_{21}, C_{22}) \geq H_2 \;. \tag{1.21}$$

In this problem, we allocate the available commodities so as to maximize consumer 1's utility subject to consumer 2 achieving a utility level of at least $H_2$. (If $H_2$ were set at $\bar{H}_2$ in our example, then problem (1.19) – (1.21) would generate point B in Figure1.1).

A third approach to generating points on AA is to solve problems of the form: choose non-negative values for $C_{ij}$, i,j=1,2

to maximize

$$U_1(C_{11}, C_{12}) \tag{1.22}$$

subject to

$$\sum_{i=1}^{2} C_{ij} - \sum_{i=1}^{2} Z_{ij} \leq 0 \;, \quad j=1,2 \tag{1.23}$$

and

$$U_2(C_{21}, C_{22}) \geq \beta\, U_1(C_{11}, C_{12}) \;. \tag{1.24}$$

This time, we maximize consumer 1's utility subject to consumer 2 achieving a utility level of at least $\beta$ times that of consumer 1. (If $\beta$ were set at 1 in our example, then problem (1.22) – (1.24) would generate point B in Figure1.1.)

The question remains of how we should set the $w_i$s in (1.10) – (1.11) or the $H_2$ in (1.19) – (1.21) or the $\beta$ in (1.22) – (1.24) so that the particular point generated on the utility possibilities frontier indicates a solution to our model (i) – (iv). How could we have proceeded with problem (1.10) – (1.11), for example, if we had not known the appropriate values ($w_1 = \frac{1}{2}$, $w_2 = \frac{1}{2}$) for the weights *a priori*? Except in very simple cases, we must adopt an iterative approach. That is, we must make an initial guess of the appropriate values for the $w_i$s; solve problem (1.10) – (1.11); use information from our solution to improve our guess of the $w_i$s; re-solve (1.10) – (1.11); improve our guess, etc. If in our numerical example, we had made as an initial guess for the weights

$$w_1^{(1)} = \frac{1}{4} \quad , \quad w_2^{(1)} = \frac{3}{4} \quad , \qquad (1.25)^6$$

then our initial solution to problem (1.10) – (1.11) would have been

$$\left(\bar{C}_1^{(1)}\right)' = (25,\ 25) \quad , \quad \left(\bar{C}_2^{(1)}\right)' = (75,\ 75) \quad , \qquad (1.26)^7$$

with the Lagrangian multipliers being

$$\left(\bar{P}^{(1)}\right)' = (0.01,\ \ 0.01) \quad . \qquad (1.27)$$

Interpreting the Lagrangian multipliers as commodity prices, this would have implied a value of

$$\left(\bar{P}^{(1)}\right)'\left(\bar{C}_1^{(1)}\right) = 0.5$$

for consumer 1's expenditure and

$$\left(\bar{P}^{(1)}\right)' Z_1 = 1.0$$

for the consumer's income. For consumer 2, expenditure and income would have been 1.5 and 1.0 respectively. These expenditure and income levels are inconsistent with condition (i). To move towards a solution to model (i) – (iv), we would have had to increase $w_1$ relative to $w_2$ and to re-solve problem (1.10) – (1.11). With a higher value for $w_1/w_2$, problem (1.10) – (1.11) would allocate more consumption to

---

6 We use the superscript (s) to denote values of variables in the s[th] iteration. For example, $w_1^{(1)}$ and $w_2^{(1)}$ are the weights in the first iteration.

7 (1.26) and (1.27) are easily established by revising (1.15) to read

$$\left.\begin{aligned} 1/(4\bar{C}_{1j}) &= \bar{P}_j, j=1,2 \\ 3/(4\bar{C}_{2j}) &= \bar{P}_j, j=1,2. \end{aligned}\right\} \qquad (1.28)$$

consumer 1 and less to consumer 2. Your exercises and readings will indicate various rules for moving the ws, Hs or βs which have proved effective in practical applications.

The final question for this section is: what are the advantages of mathematical programming as a means of solving general equilibrium models compared with working with systems of excess demand functions. Proponents of the mathematical programming approach normally emphasize dimensionality. When we work with the excess demand functions we have $n - 1$ unknowns where n is the number of commodity prices. (One price can be eliminated via normalization.) In programming problems such as (1.10) – (1.11), (1.19) – (1.21) and (1.22) – (1.24) we have to find values for $k - 1$ iterative variables where k is the number of consumers. (In problem (1.10) – (1.11), one of the ws can be eliminated via, for example, the normalization rule (1.12).) In many applied models, k (the number of consumers) is small compared to n (the number of commodities). For instance, it is quite common to treat the household sector as if it were composed of a single utility-maximizing consumer. With $k = 1$, we can solve models such as (i) – (iv) simply as mathematical programming problems without any iterative search being required. With $k = 2$, our iterative search involves no more than moving around a one-dimensional frontier such as AA in Figure1.1. Even with larger values of k, say $k = 10$, it may be easier to search in a $(k - 1)$-dimensional weight-space (or H-space or β-space) than to derive and solve a system of excess demand functions involving, perhaps, several hundred unknown commodity prices. Of course, in choosing between competing computational approaches, researchers consider many factors apart from dimensionality. For example, are the programming problems which must be solved at each step of the weight-iteration procedure of a computationally simple form? Can they be adequately approximated as linear programming problems? If so, has the local computer centre access to a cheap, user-friendly, linear programming package? If excess demand equations are used, is it necessary that they be excess demand equations for commodities? Would it be preferable to work with excess demand equations for factors? What packages are available for solving systems of non-linear equations? What modifications would be necessary before the available packages could be applied to systems of excess demand functions?

## *2 Normative Versus Positive Analysis*

Some of the literature on mathematical programming models blurs the distinction between normative and positive analysis. It is in the hope of helping you to keep the distinction clear that we provide this section.

In the previous section we showed how the simple model (i) – (iv) can be solved via mathematical programming. More generally, in the programming approach to economic modelling it is assumed that the vector X of endogenous variables either *should* be set so as to maximize or *will* maximize a function

$$f(X, Z) \tag{2.1}$$

subject to

$$g(X, Z) \leq 0 \tag{2.2}$$

where (2.2) is a set of constraints containing, for example, demand and supply balances for commodities and factors, and Z is the vector of exogenous variables. The Lagrangian multipliers (also called "shadow prices") associated with the constraints can normally be interpreted as commodity and factor prices.

Some models expressed in the form (2.1) – (2.2) are intended to be normative (prescriptive) while others are intended to be positive (descriptive). Because the distinction is important we emphasized, in the previous paragraph, the words "should" (normative models) and "will" (descriptive models).

In normative models, positive weights may appear in the objective function (2.1) on variables pertaining to economic growth and equality of income distribution. Alternatively, growth and distribution targets may be included among the constraints. We may also find policy instruments (e.g., taxes, tariffs and government spending) among the choice variables (X). In descriptive models, (1.1) is usually an aggregate utility function formed (explicitly or implicitly) as a weighted sum of the utility functions of individual consumers where the weights reflect relative spending power. Far from giving positive weight to income equality, the objective function in a descriptive model will emphasize the preferences of relatively rich consumers.

This does not mean, however, that normative analysis is precluded with a descriptive model. It means that analysis is separated into descriptive and normative stages. In the first stage, problem (2.1) – (2.2) is solved with different values for Z. The Z vector in a descriptive model will normally include policy instruments, the computations often being used to describe the effects of adopting different policies under the assumption that the economy operates as if it were maximizing consumer utility subject to resource constraints. In the second stage, the welfare implications of the results of the first stage are evaluated outside the model. This stage will usually be quite informal. The analysts may merely state that he prefers policy A to policy B because he estimates from his model that A will have the more favourable impact on income distribution.

Here we will be concerned with descriptive modelling although references are made to some normative planning models.

### *3 Goals, Reading Guide and References*

By the time you have completed your reading and finished the exercises, we hope that you will have

(1) a facility for expressing general equilibrium models as mathematical programming problems;

(2) a thorough understanding of how general equilibrium models can be solved by a sequence of mathematical programming problems;

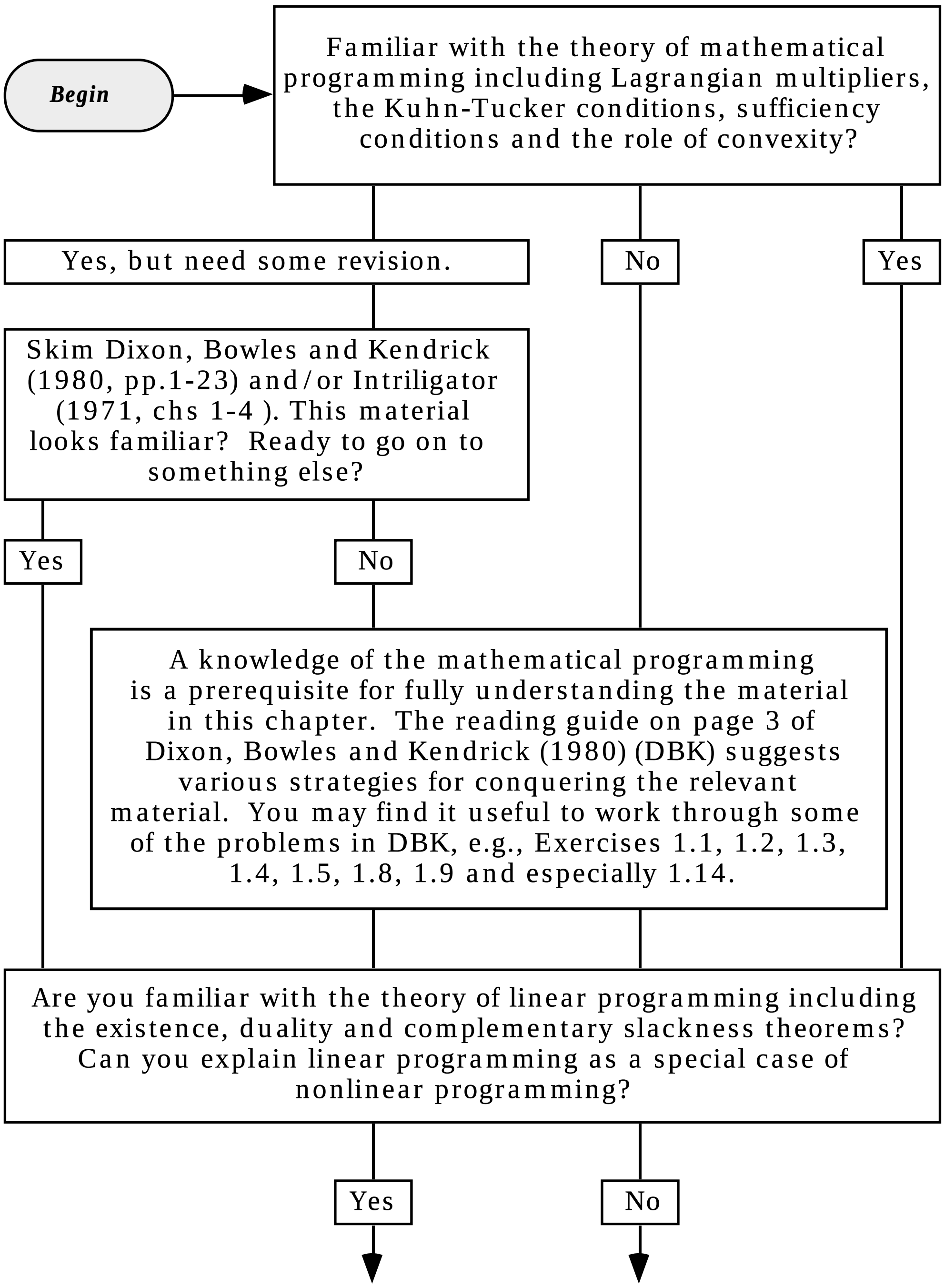
(3) a knowledge of how nonlinear mathematical programming problems with convex constraint sets and concave objective functions can be solved as linear programming problems;

(4) an explanation of the overspecialization problem both in intuitive economic terms (what diversifying phenomena are missing) and in terms of the theory of linear programming; and

(5) a familiarity with how investment, restrictions on international trade (tariffs and quotas) and taxes on commodities and factors are handled in the mathematical programming framework.

The reading guide lists some material which will help you to achieve these goals. The readings are referred to in abbreviated form. Full citations are in the reference list. The list also contains other references appearing in the chapter.

***Reading guide ****



***Reading guide (continued)***

Yes

No

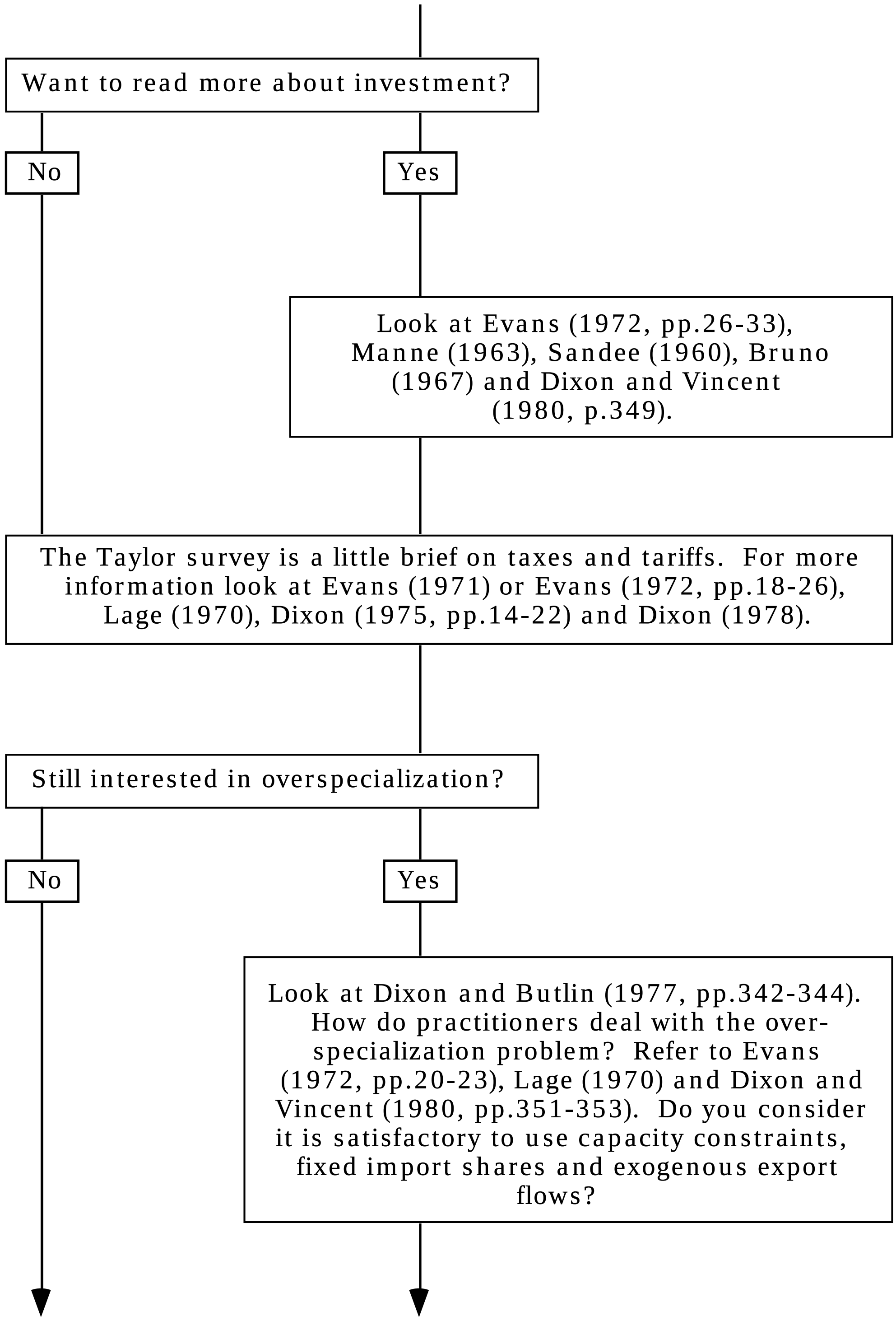
Read Intriligator (1971, pp.72-89). Pay particular attention to pp.79-86. This reference contains just the right amount of detail for our purposes. We have also provided some brief notes emphasizing the idea of basic solutions in the Appendix. These notes may be sufficient if you need just a quick reminder about linear programming.

The programming problems encountered in applied general equilibrium modeling are usually of a nonlinear variety but with concave objective functions and convex constraint sets. Approximate solutions to these nonlinear programming problems can be obtained by solving suitably chosen linear programming problems. At most computer centres, even very large linear programming problems can be solved cheaply and conveniently. It will therefore be useful for you to know how nonlinear programming problems can be converted into linear programming problems. Read Hadley (1964, pp.104-111) and/or Vajda (1961, pp.218-222).

For an elementary introduction to the idea of solving general equilibrium models by a sequence of mathematical programming problems, read Dixon (1975, pp.1-11). Earlier expositions of the relevant theory are cited there. Of particular importance is Negishi (1960). A recent development is Manne, Chao and Wilson (1980). For straightforward applications of the theory see Dixon (1975, pp.22-39), Ginsburgh and Waelbroeck (1976) and Dixon (1978).

Read Taylor (1975, pp.59-83). This reference covers the theoretical problems of handling investment, taxes and tariffs in the mathematical programming framework. It also contains a section on the overspecialization problem.

***Reading guide (continued)***



***Reading guide (continued)***

What are some of the arguments about actual economies that mathematical programming models have been used to support? Refer to the results and conclusions sections of some of the papers you have already read, e.g. Evans (1971), Evans (1972, chs 5-10), Lage (1970), Dixon and Vincent (1980) and Manne (1963). Do you find these conclusions a worthwhile payoff for the effort that has gone into building the models?

Review the list of goals given in this section. Be sure you have reached them by the time you finish your reading and problems.

***Exit***

* For full citations, see reference list for this chapter.

***References***

Baumol, W.J. (1972) *Economic Theory and Operations Analysis*, 3rd edition, Prentice-Hall.

Bruno, M. (1967) "Optimal Patterns of Trade and Development", *The Review of Economics and Statistics*, Vol. 49, November, 545-554.

Corden, W.M. (1971) *The Theory of Protection*, Clarendon Press, Oxford.

Dixon, P.B. (1975) *The Theory of Joint Maximization*, North-Holland.

Dixon, P.B. (1991) "A General Equilibrium Approach to Public Utility Pricing: Determining Prices for an Urban Water Authority", *Journal of Policy Modeling.*

Dixon, P.B. (1978) "The Computation of Economic Equilibria: A Joint Maximization Approach", *Metroeconomica*, Vol. XXIX, 173-185.

Dixon, P.B., S. Bowles and D. Kendrick (DBK) (1980) *Notes and Problems in Microeconomic Theory*, North-Holland.

Dixon, P.B. and M.W. Butlin (1977) "Notes on the Evans Model of Protection", *Economic Record*, Vol. 53 (143), June/September, 337-349.

Dixon, P.B. and D.P. Vincent (1980) "Some Economic Implications of Technical Change in Australia to 1990-91: An Illustrative Application of the SNAPSHOT Model", *Economic Record*, Vol. 56, December, 347-361.

Evans, H.D. (1971) "Effects of Protection in a General Equilibrium Framework", *Review of Economics and Statistics*, Vol. 53, May, 147-156.

Evans, H.D. (1972) *A General Equilibrium Analysis of Protection*, North Holland.

Gale, David (1960) *The Theory of Linear Economic Models*, McGraw-Hill Book Company.

Gayer, A.D., W.W. Rostow and A.J. Schwartz (1953) *The Growth and Fluctuation of the British Economy*, 1790-1850, 2 volumes, Clarendon, Oxford.

Ginsburgh, V. and J. Waelbroeck (1976) "Computational experience with a large general equilibrium model", pp. 257-269 in J. Los and W. Los (eds), *Computing Equilibria: How and Why*, North-Holland Publishing Company.

Hadley, G. (1964) *Non-Linear and Dynamic Programming*, Addison Wesley.

Hazari, B.R. (1978) *The Pure Theory of International Trade and Distortions*, Croom-Helm Publishing Company.

Intriligator, M.D. (1971) *Mathematical Optimization and Economic Theory*, Prentice-Hall.

Lage, G.M. (1970) "A Linear Programming Analysis of Tariff Protection", *Western Economic Journal*, Vol. viii, 167-185.

Leontief, W.W. (1978) "Issues of the coming years", *Economic Impact*, No. 24 (4), 70-76.

Lipsey, R.G. and K. Lancaster (1957) "The General Theory of the Second Best", *Review of Economic Studies*, Vol. 24, February, 11-32.

Manne, A.S. (1963) "Key Sectors of the Mexican Economy 1960-1970", pp. 379-400 in A.S. Manne and H.M. Markowitz (eds), *Studies in Process Analysis*, Wiley.

Manne, A.S., Hung-Po Chao and R. Wilson (1980) "Computation of Competitive Equilibria by a Sequence of Linear Programs", *Econometrica*, Vol. 48 (7), November, 1595-1616.

Negishi, T. (1960) "Welfare Economics and the Existence of an Equilibrium for a Competitive Economy", *Metroeconomica*, Vol. XII, 92-97.

Samuelson, P.A. (1950) "Evaluation of Real National Income", *Oxford Economic Papers*, Vol. 2 (NS), January, 1-29.

Sandee, J. (1960) *A Long-term Planning Model for India*, Asia Publishing House, New York and Statistical Publishing Company, Calcutta.

Sauvy, Alfred (1969) *General Theory of Population*, Weidenfeld and Nicholson, London

Taylor, L. (1975) "Theoretical Foundations and Technical Implications", pp. 33-109 in C.R. Blitzer, P.B. Clark and L.Taylor (eds), *Economy-Wide Models and Development Planning*, Oxford University Press.

Vajda, S. (1961) *Mathematical Programming*, Addison-Wesley Publishing Company.

### *Exercise 1 The implications of technical change in a wine-cloth economy*

This is an easy warm-up exercise. It uses a very elementary model to introduce a few important general equilibrium ideas without algebraic complications. It is organized in four sections. The first describes the net annual production possibilities set, i.e., what *can* be produced with existing techniques and resources in one year. The second introduces consumer preferences to determine what *will* be produced. In the third we discuss prices and real wages. In the last section we are ready to derive the implications of a change in production techniques.

Table E1.1

*Current Production Techniques: Input-Output Coefficients*

| | *Outputs* | |
|---|---|---|
| *Inputs* | Wine (1 gallon) | Cloth (1 yard) |
| Wine | *nil* | *nil* |
| Cloth | 0.2 yards | *nil* |
| Labour | 1 hour | 1 hour |

*Section 1: What can be produced?*

Consider a society which produces just two products, wine and cloth. The techniques currently in use for the production of these products are described by input-output coefficients in Table E1.1. The output of one gallon of wine requires an input of 0.2 yards of cloth and one hour of labour. The output of one yard of cloth requires an input of just one hour of labour.

We assume that the society's resource endowment for a year is 100 labour hours. Given this resource endowment and the production techniques set out in Table E1.1, we can derive the society's net annual production possibilities set. This is shown graphically in Figure E1.1. It can be constructed by doing a few calculations. For example,

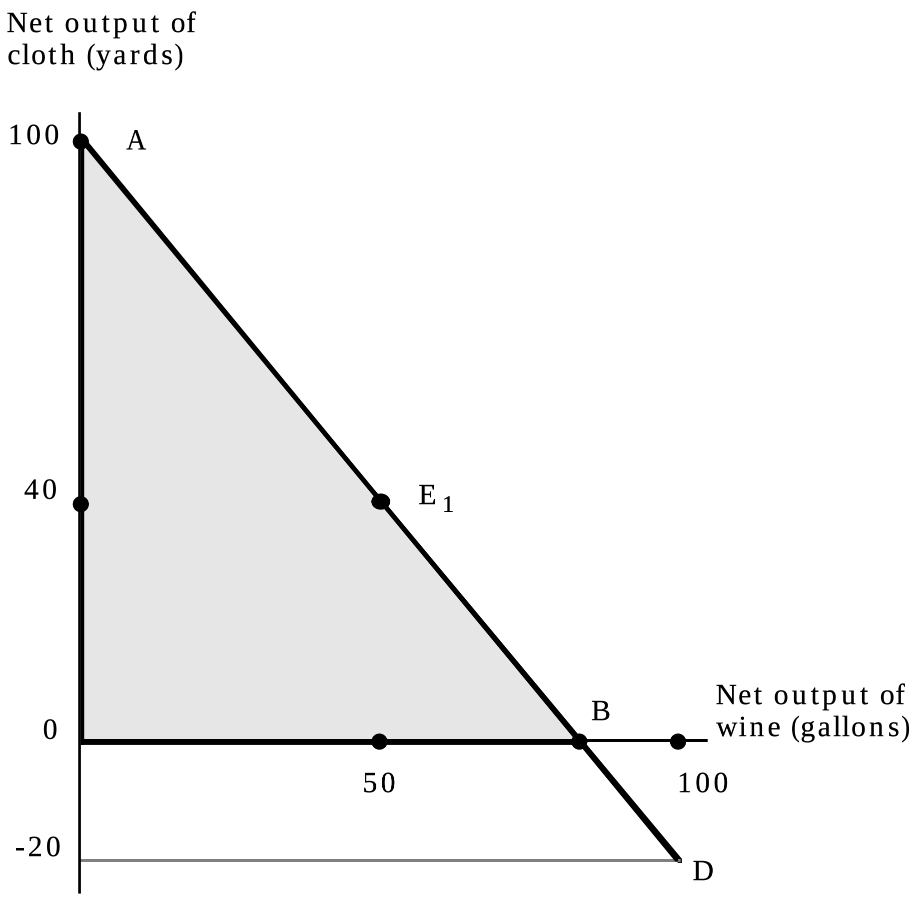



Figure E1.1 Net annual production possibilities

Given the production techniques in Table E1.1 and a resource endowment of 100 labour hours, our society can produce for final use any combination of wine and cloth shown in the triangle 0AB.

if 50 labour hours were devoted to wine and 50 to cloth, then society's net annual output would be 50 gallons and 40 yards. Notice that the gross output of cloth is 50 yards, but that 10 yards are used up in wine production. We have marked the point 50 gallons and 40 yards as $E_1$ in Figure E1.1. Now consider the case in which our society allocates all its resources (100 labour hours) to cloth production. Net annual output would be 100 yards of cloth and no wine (point A in Figure E1.1). On the other hand, if all the labour were devoted to wine, we would end up with 100 gallons, but we would have a deficit of 20 yards of cloth (point D in Figure E1.1). Perhaps the deficit could be made up by drawing on accumulated stocks or through importing. But for simplicity we will assume that there are no accumulated stocks and that there is no international trade. Thus, the net annual production possibilities available to our society are restricted to the shaded area 0AB in Figure E1.1.

(a) Can our society produce a net annual output of 40 gallons of wine and 60 yards of cloth?

(b) Can our society produce a net annual output of 40 gallons of wine and 30 yards of cloth?

(c) Consider the production techniques shown in Table E1.2. The only change from those in Table E1.1 is in the labour input coefficient for cloth. Technical progress has taken place which allows a yard of cloth to be produced with only 0.5 hours of labour input rather than one hour. Assume that the society's resource endowment remains at 100 labour hours per year. Can you construct the new net annual production possibilities set?

Table E1.2

*Production Techniques after an Improvement in the Technique for Producing Cloth*

| | Wine (1 gallon) | Cloth (1 yard) |
|---|---|---|
| Wine | *nil* | *nil* |
| Cloth | 0.2 yards | *nil* |
| Labour | 1 hour | 0.5 hours |

*Section 2: What will be produced?*

Having constructed society's net annual production possibilities set and illustrated it in Figure E1.1, our next task is to determine which point in the set will be chosen. This will depend on (i) the level of employment and (ii) consumer preferences for wine and cloth.

Let us make the assumption that our society achieves full employment, i.e., all of the 100 labour hours are used in production. This contentious assumption is discussed in some detail in the final part of this exercise. If we accept the full employment assumption, then we can restrict our search for the actual net production point to the frontier, AB, of the net annual production possibilities set. It is only on the frontier that we have full employment.

Which point will be chosen on the frontier, AB? This will depend on what society wants to consume. Let us consider the simplest possible case by assuming that our society always consumes wine and cloth in fixed proportions: 5 gallons of wine to 4 yards of cloth. In terms of Figure E1.2, consumption will occur somewhere along the line 0C. In fact, in view of our full employment assumption, we can see that net production and consumption of wine and cloth will be 50 gallons and 40 yards (point $E_1$ in Figure E1.2).

(d) At $E_1$ how many labour hours will be used in the production of wine? How many will be used in the production of cloth?

(e) Continue to assume that total employment is 100 labour hours and that wine and cloth are consumed in the ratio of 5 gallons to 4 yards. If the production technique for cloth improves to that shown in Table E1.2, what will be the new levels for net production and consumption of wine and cloth? How many labour hours will be used in wine production? How many in cloth production?

*Section 3: Commodity prices and real wages*

At this stage we have come a long way. Starting from a description of production techniques and consumer preferences, we have found out what our society will produce and consume, and what employment will be in each industry. We can also determine commodity prices and the real hourly wage rate.[8]

Suppose that the nominal wage rate is $1 per hour. Then under the production techniques shown in Table E1.1, the price of cloth would be $1 per yard. This is because it takes one hour of labour to make a yard of cloth. The price of a gallon of wine would be $1.2, i.e., the cost of one hour of labour plus the cost of 0.2 yards of cloth. The hourly wage ($1) would be just sufficient to buy a commodity bundle containing 0.5 gallons of wine and 0.4 yards of cloth.

(f) If the wage rate were $10 per hour, what would be the prices of wine and cloth? Would the wage for an hour's labour still buy a commodity bundle containing 0.5 gallons of wine and 0.4 yards of cloth? In determining the *real* hourly wage rate, does it make any difference whether we assume the nominal hourly wage is $1 or $10?

(g) Assume that the wage rate is $1 per hour and that the production techniques are those shown in Table E1.2. What will be the prices of wine and cloth? Check that the wage for an hour's labour can now buy a commodity bundle containing 0.67 gallons of wine and 0.53 yards of cloth.

*Section 4: The effects of a change in production techniques*

In Table E1.3 we have listed everything that we have found out about the economy of our wine-cloth society. Column I shows commodity outputs, employment in each industry, commodity prices and the real wage rate in the initial situation (i.e., when the production techniques are those in Table E1.1). Column II shows the corresponding results when the production techniques are those in Table E1.2. By comparing columns I and II, we can see the economy-wide effects of the improvement in the production technique for cloth. The halving of the labour input coefficient to cloth production has allowed consumption and net production of both wine and cloth to increase by $33\frac{1}{3}$ per cent, real wage rates to increase by $33\frac{1}{3}$ per cent,

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8 The real hourly wage rate is measured by a quantity of commodities that can be purchased in return for one hour's labour.

Table E1.3

*The Wine-Cloth Economy before and after the Improvement in the Technique for Producing Cloth*

| | I. Before | II. After |
|---|---|---|
| *Net annual output and consumption* | | |
| Wine | 50 gallons | $66\frac{2}{3}$ gallons |
| Cloth | 40 yards | $53\frac{1}{3}$ yards |
| *Employment* | | |
| Wine | 50 hours | $66\frac{2}{3}$ hours |
| Cloth | 50 hours | $33\frac{1}{3}$ hours |
| *Prices (assuming that the wage rate is $1 per hour)* | | |
| Wine | $1.2 per gallon | $1.1 per gallon |
| Cloth | $1.0 per yard | $0.5 per yard |
| *Real wage rate* | | |
| The wage for one hour's labour buys | 0.5 gallons plus 0.4 yards | 0.67 gallons plus 0.53 yards |

the price of cloth to fall sharply relative to that of wine and $16\frac{2}{3}$ per cent of the labour force to be reallocated from the cloth industry to the wine industry.

This analysis is quite similar to that used by economists concerned with quantifying the effects of technical change in the real world. For example, in their study of the Australian economy, Dixon and Vincent (1980) assembled two tables of input-output coefficients, one showing production techniques as they were in 1971/72 and the other showing the production techniques forecast for 1990/91.[9] They then made some comparisons. Their central computation was designed to answer the following question: how much difference do the projected changes in production techniques make to one's picture of how the economy will be in 1990/91. In terms of Figures E1.2 and E1.3, Dixon and Vincent computed the points $E_1$ and $E_2$, where $E_1$ refers to the levels which would be achieved in 1990/91 for commodity outputs, prices, real wages, etc., if production techniques remained as they were in 1971/72 and $E_2$ refers to the situation which will emerge if production techniques are consistent with the forecasts. The comparison between $E_1$ and $E_2$ was, therefore, the basis for a discussion of the implications of technical change.

Dixon and Vincent had to consider many details which were not included in our wine-cloth economy. They divided the economy into 109 sectors, rather than 2. They included capital, not just labour as a primary factor of production and they divided labour into 9 occupational groups. They considered the role of investment, not just consumption. They allowed for international trade, government expenditure and numerous taxes, tariffs and subsidies. Nevertheless, in essence, their approach consisted of the steps outlined in this exercise: (i) the derivation of alternative net annual production possibilities sets corresponding to alternative assumptions about production techniques, and (ii) the imposition of the full employment assumption and the consideration of consumer preferences leading to the calculation of the net production points.

What did Dixon and Vincent conclude from their study? Given the preliminary nature of their work and a number of deficiencies which they were careful to emphasize, they were cautious. They did, however, offer the following:

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9 The forecasts were based on work done by two agencies sponsored by the Australian Government: the Bureau of Industry Economics (BIE) and the IMPACT Project. The BIE selected industries which appeared to be undergoing rapid technical change and asked industry experts to forecast the future input-output coefficients. Where expert opinions were not available, forecasts based on trend projections were prepared by the IMPACT Project.

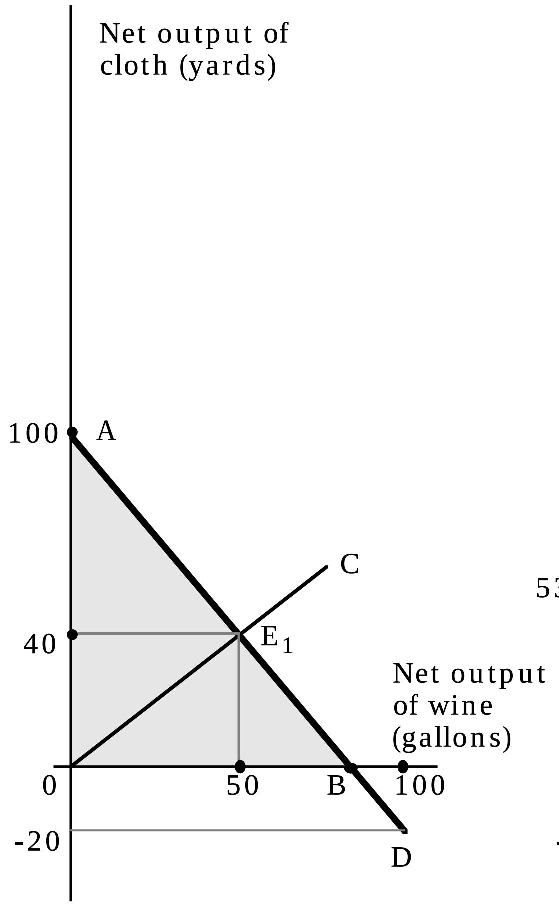


Figure E1.2 Net annual production possibilities under the initial production techniques

Apart from the addition of the consumption line, 0C, this figure is the same as Figure E1.1.

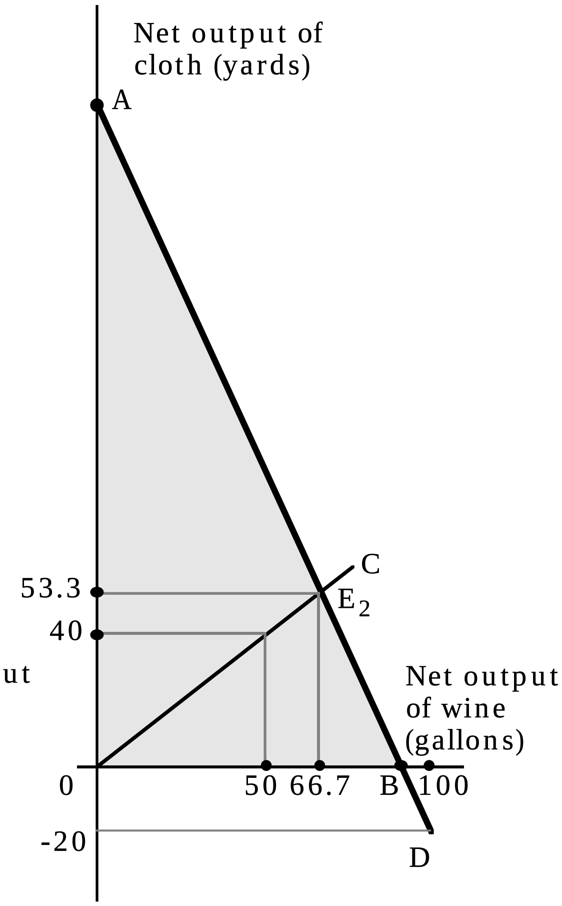


Figure E1.3 Net annual production possibilities after the improvement in the technique for producing cloth

> "The overwhelming impression from Table 4.6 [reproduced here as Table E1.4] is that the occupational composition of the workforce at the 9-order level in 1990/91 is unlikely to be radically different from that in 1971/72 and that it will be determined largely independently of technical change. Certainly, the present simulations do not pinpoint any likely difficulties in the areas of labour mobility and manpower training." [Dixon and Vincent (1980, p. 358)]

and

> "Subject to the qualifications expressed throughout the paper, our results indicate that rapid technical progress is particularly important for the future well-being of those Australian industries which are closely connected with international trade. At the macro level, our results support the view that technical progress is vital for securing increased GDP, increased consumption and higher real wages. Technical progress may also affect macroeconomic management. In the absence of technical progress, we found that the 'full-employment' level of real wages would decline. Under such conditions, it is difficult to imagine that Australia could achieve even a tolerable approximation to full employment." [Dixon and Vincent (1980, p. 359)]

(h) In Table E1.4, you will see references to innovative and Luddite economies. Who were the Luddites?

The calculations by Dixon and Vincent and our own analysis of the wine-cloth economy present technical change in a favourable light. Only its role as a source of increased material welfare is emphasized. But this is not the aspect of technical change which has always been emphasized in popular discussions. Sometimes, the principal concern has been with job replacement. Newspapers frequently report fears expressed by various groups in the community concerning the employment effects of new machines: word-processors, automatic bank tellers, point-of-sale terminals, vending machines and robots.

Let us re-examine our wine-cloth story from the point of view of the employment implications of technical change. The critical assumption in the story is that technical change is not an important determinant of the aggregate level of employment. It is assumed that aggregate employment is 100 labour hours both before and after the improvement in the production technique for cloth. There is no need to assume that

Table E1.4

*Occupational Shares in the Workforce**

| | | I<br>Actual<br>1971/72<br>Economy | II<br>1990/91<br>Innovative<br>Economy†,# | III<br>1990/91<br>Luddite<br>Economy†,# |
|---|---|---|---|---|
| 1. | Professional White Collar | 3.3 | 3.9<br>(3.1) | 4.0<br>(3.2) |
| 2. | Skilled White Collar | 12.8 | 14.3<br>(2.7) | 13.6<br>(2.5) |
| 3. | Semi- and Unskilled White Collar | 26.9 | 30.5<br>(2.9) | 29.0<br>(2.6) |
| 4. | Skilled Blue Collar (Metal & Electrical) | 10.9 | 9.0<br>(1.2) | 9.1<br>(1.3) |
| 5. | Skilled Blue Collar (Building) | 5.1 | 3.9<br>(0.8) | 4.1<br>(1.0) |
| 6. | Skilled Blue Collar (Other) | 2.6 | 2.7<br>(2.5) | 2.7<br>(2.5) |
| 7. | Semi- and Unskilled (Blue Collar) | 32.0 | 30.5<br>(1.9) | 30.6<br>(1.9) |
| 8. | Rural Workers | 4.8 | 3.6<br>(0.5) | 5.1<br>(2.4) |
| 9. | Armed Services | 1.6 | 1.6<br>(2.2) | 1.8<br>(2.7) |
| | | 100.0 | 100.0<br>(2.2) | 100.0<br>(2.2) |

* Source: Dixon and Vincent (1980, p. 359).

† Figures in parentheses show annual percentage growth rates over the period 1971/72 to 1990/91. For example, professional white collar employment grows at an average annual rate of 3.1 per cent on the path to the Innovative Economy of 1990/91.

# Innovative and Luddite were the labels used in Dixon and Vincent (1980). Luddite refers to the calculations for 1990/91 with the input-output coefficients set at their 1971/72 values (the $E_1$ results in Figure E1.2). Innovative refers to the calculations where the input-output coefficients were set at the values forecast for 1990/91 (the $E_2$ results in Figure E1.3). The differences between columns II and III were interpreted as being attributable to technical change.

employment of 100 labour hours is literally full employment. Perhaps 105 hours of labour are available. Our assumption is that 5 per cent unemployment is just as likely with technical progress as without it.[10]

This assumption should not be too surprising to readers with some knowledge of conventional macroeconomic theory. That theory stresses demand management, fiscal and monetary policy and the real wage rate in relation to labour productivity as the major determinants of aggregate employment. The rate of technical change rarely rates even a mention. This will not be very reassuring to readers who are sceptical about conventional economic theory. They will want us to spell out the process by which workers, displaced by technical change, will find new jobs.

In terms of our wine-cloth economy, the problem is to explain the transition from $E_1$ to $E_2$ (Figures E1.2 and E1.3). Starting at $E_1$, the halving of the labour input coefficient for cloth will mean that only 25 labour hours (rather than 50) are required in the industry. Cloth now will be cheaper and the real incomes of employed workers will expand. These workers will demand more wine and cloth, thus providing employment for the previously displaced workers. This will set us on the happy path to $E_2$.

What if capitalists prevent the reduction in the real price of cloth by taking an increase in profits? But don't capitalists consume wine and cloth too? Perhaps not, perhaps capitalists spend on imported luxuries and overseas holidays. But what will the foreigners do with the domestic dollars they receive from the capitalists? They will buy our wine and cloth! But what happens if everyone has had enough wine and cloth? This would be a blissful state — we could simply do less work. Unfortunately a state in which all our material wants are satisfied seems very far away, even in the world's wealthiest countries.

What about adjustment problems along the path from $E_1$ to $E_2$? Recall that the shift from $E_1$ to $E_2$ involved the transfer of $16\frac{2}{3}$ per cent of the labour force out of cloth and into wine. What if the skills required of wine workers differ from those of cloth workers? Then might not

10 This assumption may be overly generous to the situation with no technical progress. With no technical progress there is unlikely to be scope for increases in real wages without reductions in employment.

the move from $E_1$ to $E_2$ cause excessive periods of unemployment for surplus cloth workers? Certainly this is a possibility. It is important, therefore, in comparing $E_1$ and $E_2$ to consider the feasibility of the implied rates of shift of resources between different activities. This is what Dixon and Vincent did in their analysis of the implications of technical change to 1990/91. For example, on examining Table E1.4, they concluded that technical change to 1990/91 could be accommodated without rapid transfers of labour between the nine broadly defined occupational groups. It is possible, however, that technical change to 1990/91 may render redundant certain very specific skills. This does not necessarily imply any serious difficulties. In many countries, workers exhibit a high degree of occupational mobility.

We conclude this exercise with two stories about horses, and a question.

*Horse story number one*[11]

Maynard, the employer, and his worker, Milton, produce 20 bushels of wheat per year from 5 acres of land. Maynard pays Milton a wage of 10 bushels and retains a profit of 10 bushels for himself.

One day, Maynard makes a remarkable technical improvement. He captures and trains a horse. Using the horse, Maynard can produce 20 bushels of wheat per year without Milton's help. Since the horse consumes only 7 bushels of wheat, Maynard sacks Milton and lives happily ever after consuming 13 bushels of wheat per year.

But what of poor Milton? He leaves the farm and starves to death.

*Horse story number two*[12]

Anyone who doesn't believe in the possibility of permanent unemployment arising from technical change should think about what happened to employment prospects for horses at the beginning of this century.

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11 This story is adapted from Sauvy (1969, p. 113).

12 This story is adapted from Leontief (1978).

(i) Have either of these horse stories any relevance to the analysis of the implications of technical change in a modern economy? Both imply the possibility of an unhappy outcome from technical change. What are the key differences in the assumptions underlying our wine-cloth analysis and the assumptions underlying the horse stories?

***Answer to Exercise 1***

**(a)** No. In terms of Figure E1.1, the point 40 gallons, 60 yards lies outside the triangle 0AB. Given the production techniques in Table E1.1, it would take a gross output of 40 gallons and 68 yards to achieve a net output of 40 gallons and 60 yards. Thus, 108 hours of labour would be required. Only 100 hours are available.

**(b)** Yes. In terms of Figure E1.1, the point 40 gallons, 30 yards is inside the triangle 0AB. Given the production techniques in Table E1.1, it would take a gross output of 40 gallons and 38 yards to achieve a net output of 40 gallons and 30 yards. Thus, 78 hours of labour would be required. This is available.

**(c)** The new net annual production possibilities set is shown in Figure E1.3 as the triangle 0AB. It can be constructed by considering the net outputs which would emerge as we vary the allocation of labour between the production of our two commodities. For example, if all the 100 labour hours were devoted to cloth production, then we would obtain 200 yards of cloth and no wine. Hence point A in Figure E1.3 is part of the new net annual production possibilities set. If $66\frac{2}{3}$ hours of labour were devoted to wine and $33\frac{1}{3}$ to cloth, then net production would be $66\frac{2}{3}$ gallons of wine and $53\frac{1}{3}$ yards of cloth, point $E_2$ in Figure E1.3. (Notice that the use of $33\frac{1}{3}$ labour hours in cloth generates a gross output of $66\frac{2}{3}$ yards, but that the wine production uses up $13\frac{1}{3}$ (= $0.2 \times 66\frac{2}{3}$ yards.) If 100 labour hours were devoted to wine production, then we would obtain 100 gallons. There would, however, be a deficit of 20 yards of cloth. (See point D in Figure E1.3.) Because we rule out both international trade and the existence of accumulated stocks, deficits are not possible. Hence the net annual production possibilities are confined to the triangle 0AB in Figure E1.3.

Be sure to compare the new net annual production possibilities set with the old one. As can be seen by looking at Figures E1.2 and E1.3, technical progress in the cloth industry leads to an expansion of the possibilities set.

**(d)** At $E_1$, the *gross* outputs are 50 gallons of wine and 50 yards of cloth. Hence 50 labour hours are used in wine production and 50 in cloth production.

**(e)** In Figure E1.3, the consumption line 0C crosses the frontier of the net annual production possibilities set at $E_2$. The levels for net production and consumption of wine and cloth are $66\frac{2}{3}$ gallons and $53\frac{1}{3}$ yards. Employment is $66\frac{2}{3}$ hours in wine and $33\frac{1}{3}$ hours in cloth.

**(f)** If the wage rate were \$10 per hour, the price of cloth would be \$10 per yard and the price of wine would be \$12 per gallon. The real wage rate would be unaffected by an increase in the wage rate from \$1 per hour to \$10 per hour. In both cases, the wage for an hour's labour would buy a bundle of commodities containing 0.4 yards of cloth and 0.5 gallons of wine.

**(g)** The price of cloth will be \$0.5 per yard and the price of wine will be \$1.1 per gallon. At these prices, a commodity bundle containing 0.67 gallons of wine and 0.53 yards of cloth would cost \$1, i.e., the wage for one hour of labour would buy a bundle containing 0.67 gallons of wine and 0.53 yards of cloth.

**(h)** The Luddites were organized bands of English workmen who in 1811-12 destroyed stocking frames, steam power looms and shearing machines in various centres of the British textile industry. A popular belief at the time was that these recently introduced labour-saving machines were a cause of low real wages and high unemployment. Luddite activity again broke out in 1816.

It is doubtful that technical advances taking place in the British textile industry in the early 19th century had anything to do with the particularly miserable position of British workers in 1811-12 and 1816. In both these years the British economy was in a state of recession arising from poor harvests and high food prices. By blaming technical progress (machines), the workers appear to have misidentified the source of their problems. See, for example, Gayer, Rostow and Schwartz (1953, pp. 135-7).

**(i)** The key differences between the assumptions underlying the horse stories and those underlying the wine-cloth story concern human adaptability to change. In the wine-cloth story, the displaced cloth workers can move into wine production. By contrast, horse story number one depicts the displaced worker, Milton, as having no viable alternative to working for Maynard. One wonders why Milton does not capture a horse and work some land of his own. Perhaps society has advanced to the stage where all the arable land is occupied. But then it is surprising that Milton does not go to a town and work in an urban occupation. Horse story number two also depicts the possibility of the displaced workers being left with nothing to do, the displaced workers this time being horses. We should note, however, that the horses which lost their jobs at the turn of the century had a very limited range of skills compared with human workers.

***Exercise 2 A single-consumer linear economy***

Consider an economy in which an equilibrium is a list of non-negative vectors and scalars {p, γ, x} satisfying

$$\left.\begin{array}{c}\gamma a \le z + Ax \\ p'(\gamma a - z - Ax) = 0\end{array}\right\} \qquad \text{(E 2.1)}$$

$$\left.\begin{array}{c}p'A \le 0 \\ p'Ax = 0\end{array}\right\} \qquad \text{(E2.2)}$$

and

$$p'a = 1 \quad , \qquad \text{(E2.3)}$$

where p is the $n \times 1$ vector of commodity prices, $\gamma$ is a scalar indicating the number of commodity bundles consumed and x is an $m \times 1$ vector of production activity levels. The exogenous variables are a, the $n \times 1$ vector giving the commodity composition of the consumption bundle; z, the $n \times 1$ vector giving the economy's resource endowments and A, the $n \times m$ production technology matrix. The $ij^{th}$ element of A is the input (if negative) or output of good i per unit of activity j.

Condition (E2.1) says that demand is less than or equal to supply and that commodities in excess supply have a price of zero. Condition (E2.2) implies that no activity is operated at a positive profit and that activities involving losses are operated at the zero level. Condition (E2.3) fixes the absolute price level. This can be done in other ways. For example, in the last exercise, it was the wage rate which we fixed at one rather than the value of the consumption bundle.

(a) Assume that

$$a = \begin{pmatrix} 5 \\ 4 \\ 0 \end{pmatrix}, \quad z = \begin{pmatrix} 0 \\ 0 \\ 100 \end{pmatrix} \text{ and } A = \begin{pmatrix} 1.0 & 0.0 \\ -0.2 & 1.0 \\ -1.0 & -1.0 \end{pmatrix}.$$

What are the equilibrium values for p, $\gamma$ and x? If we let the three commodities be wine, cloth and labour, then (apart from the choice for the absolute price level) the present model is the same as that in Exercise 1 before the improvement in the technology for making cloth.

(b) Assume that a new technique for producing commodity 2 becomes available. The economy's production technology matrix is now

$$A = \begin{pmatrix} 1.0 & 0.0 & -0.1 \\ -0.2 & 1.0 & 1.1 \\ -1.0 & -1.0 & -1.0 \end{pmatrix}$$

Will the new technique be used?

(c) In special cases where the numbers of commodities, n, and activities, m, are small, it is possible to solve the model (E2.1) – (E2.3) by elementary graphical and/or algebraic methods. In empirically interesting cases, where n and m may be large, these methods are not adequate. Write down a linear programming problem which would be a suitable vehicle for solving (E2.1) – (E2.3).

***Answer to Exercise 2***

**(a)** In this economy, activity 1 is the only method for producing commodity 1 and activity 2 is the only method for producing commodity 2. It is clear, therefore, that both activities must be operated at positive levels. Thus, we can find the equilibrium prices from

$$(p_1, p_2, p_3) \begin{pmatrix} 1.0 & 0.0 \\ -0.2 & 1.0 \\ -1.0 & -1.0 \end{pmatrix} = (0,0)$$

and

$$5p_1 + 4p_2 = 1 \quad .$$

This gives

$$(p_1, p_2, p_3) = (0.12, 0.10, 0.10) \quad .$$

Because all prices are positive, we know that the market clearing conditions hold as equalities. Thus, we can determine $\gamma$, $x_1$ and $x_2$ from

$$\gamma \begin{pmatrix} 5 \\ 4 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 100 \end{pmatrix} + \begin{pmatrix} 1.0 & 0.0 \\ -0.2 & 1.0 \\ -1.0 & -1.0 \end{pmatrix} \begin{pmatrix} x_1 \\ \\ x_2 \end{pmatrix} \quad ,$$

that is

$$\begin{pmatrix} -1.0 & 0.0 & 5 \\ 0.2 & -1.0 & 4 \\ 1.0 & 1.0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \gamma \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 100 \end{pmatrix} \quad . \qquad \text{(E2.4)}$$

On solving equation (E2.4) we obtain

$$x_1 = 50, \quad x_2 = 50 \quad \text{and} \quad \gamma = 10 \quad .$$

**(b)** Because activity 1 continues to be the only method for producing commodity 1, we may assume that it is still operated at a positive level. If the new activity (activity 3) were also operated at a positive level, the commodity prices would satisfy

$$(p_1, p_2, p_3) \begin{pmatrix} 1.0 & -0.1 \\ -0.2 & 1.1 \\ -1.0 & -1.0 \end{pmatrix} = (0, \ 0)$$

and

$$5p_1 + 4p_2 = 1 \quad ,$$

giving

$$(p_1, p_2, p_3) = (0.1193, \ 0.1009, \ 0.0991) \quad . \qquad \text{(E2.5)}$$

At these prices, the profit per unit of activity 2 is

$$(0.1193,\ 0.1009,\ 0.0991)\begin{pmatrix}0.0\\1.0\\-1.0\end{pmatrix} = 0.0018 > 0 \quad .$$

Thus the prices in (E2.5) are not consistent with (E2.2). We may conclude that the new technique will not be introduced and that the initial equilibrium will not be disturbed. (Just to be safe, check that at the initial prices, the new activity makes non-positive profits.)

**(c)** Consider the problem of choosing non-negative values for $\gamma$ and x to

$$\left.\begin{array}{ll} & \text{maximize } \gamma \\ \text{subject to} & \\ & \gamma a \le z + Ax \end{array}\right\} \qquad \text{(E2.6)}$$

If $(\bar{\gamma}, \bar{x})$ is a solution to this problem, then there exists $\bar{p} \ge 0$, such that[13]

$$1 - \bar{p}'a = 0 \quad ,^{14}$$

$$\bar{p}'A \le 0 \quad ,$$

$$\bar{p}'A\bar{x} = 0 \quad ,$$

---

13 Are you having trouble remembering how to write the conditions for a solution of (E2.6)? The Lagrangian for this problem is

$$L(\gamma, x, p) = \gamma - p'(\gamma a - z - Ax) \quad .$$

If $(\bar{\gamma}, \bar{x})$ is a solution to (E2.6), then there exists $\bar{p} \ge 0$ such that

$$\frac{\partial L}{\partial \gamma} \le 0 \ , \quad \gamma\frac{\partial L}{\partial \gamma} = 0 \ ,$$

$$\left[\frac{\partial L}{\partial x}\right]' \le 0 \ , \quad \left[\frac{\partial L}{\partial x}\right]' x = 0 \ ,$$

$$\left[\frac{\partial L}{\partial p}\right] \ge 0 \ , \quad p'\left[\frac{\partial L}{\partial p}\right] = 0 \ ,$$

where all these expressions are evaluated at $(\bar{\gamma}, \bar{x}, \bar{p})$. Need more help? Try some of the reading on mathematical programming suggested in the reading guide.

14 We assume that $\bar{\gamma} > 0$. Thus we can write this first condition as an equality.

$$\bar{\gamma} a \leq z + A\bar{x}$$

and

$$\bar{p}'(\bar{\gamma} a - z - A\bar{x}) = 0 \quad .$$

Thus solutions for models of the form (E2.1) – (E2.3) may be computed by solving linear programming problems of the form (E2.6). The solution $(\bar{\gamma}, \bar{x})$ to (E2.6), together with the associated Lagrangian multipliers or dual solution $(\bar{p})$, satisfies the conditions (E2.1) – (E2.3).

***Exercise 3 The utility possibilities frontier***

When we move from single-consumer to multiple-consumer models, the utility possibilities frontier becomes a useful concept. As is apparent from subsequent exercises, it is sometimes convenient to solve multiple-consumer models by searching the utility possibilities frontier.

In the context of an r-consumer model with a given specification of resource endowments, production technologies and external trading opportunities, the vector of utility levels $U \equiv (U_1, U_2, \ldots, U_r)$ is a point on the utility possibilities frontier if and only if U is feasible and

$$\bar{U}_i < U_i \quad \text{for at least one i}$$

if $\bar{U} \equiv (\bar{U}_1, \ldots, \bar{U}_r)$ is a feasible utility vector different from U. (U and $\bar{U}$ are feasible if it is possible to generate the consumption levels required to support them.)

In some cases, we can describe the utility possibilities frontier by an equation of the form

$$f(U_1, U_2, \ldots, U_r) = 0 \qquad \text{(E3.1)}$$

where $(U_1, \ldots, U_r)$ is a point on the utility possibilities frontier if and only if it satisfies (E3.1).

(a) Consider a two-consumer, two-commodity pure exchange model (no production) where the utility functions for the two consumers are

$$U_i = (C_{i1})^{1/2} (C_{i2})^{1/2} \quad , \quad i = 1,2 \qquad \text{(E3.2)}$$

and where $C_{ij}$ is the consumption of good j by consumer i. Assume that the consumers' endowment vectors are

$$Z_1' = (100,\ 0)$$

and

$$Z_2' = (0,\ 100) \quad .$$

Derive the equation for the utility possibilities frontier and provide a sketch of the utility possibilities set.

(b) Answer question (a) with (E3.2) replaced by

$$U_i = (C_{i1})^{1/4} (C_{i2})^{1/4} \quad , \quad i=1,2 \quad . \tag{E3.3}$$

(c) Answer question (a) with (E3.2) replaced by

$$U_1 = (C_{11})^{1/4} (C_{12})^{1/4} \tag{E3.4}$$

and

$$U_2 = (C_{21})^{1/2} (C_{22})^{1/2} \quad . \tag{E3.5}$$

(d) Answer question (a) with (E3.2) replaced by

$$U_i = \ln(C_{i1}) + \ln(C_{i2}) \quad , \quad i=1,2 \quad . \tag{E3.6}$$

(e) Show that for an r-consumer, pure exchange model, the utility possibilities set is convex if the utility functions are concave.

(f) Generalize part (e). Assume that the net production possibilities set is convex. Show that the utility possibilities set is convex if the utility functions are concave.

***Answer to Exercise 3***

**(a)** Let $(U_1, U_2)$ be a point on the utility possibilities frontier with $C_1' \equiv (C_{11}, C_{12})$ and $C_2' \equiv (C_{21}, C_{22})$ being the underlying consumption vectors. Then

$$U_1 = (C_{11})^{1/2} (C_{12})^{1/2} \ , \tag{E3.7}$$

$$U_2 = (C_{21})^{1/2} (C_{22})^{1/2} \ , \tag{E3.8}$$

$$C_{11} + C_{21} = Z_{11} + Z_{21} = 100 \tag{E3.9}$$

and

$$C_{12} + C_{22} = Z_{12} + Z_{22} = 100 \quad . \qquad \text{(E3.10)}$$

We also know that the marginal rate of substitution between goods 1 and 2 will be the same for consumer 1 as for consumer 2. If this condition were not satisfied, it would be possible to raise the utility levels of both consumers by reallocating commodities between them. Thus,

$$\frac{\partial U_1(C_1)}{\partial C_{11}} \Big/ \frac{\partial U_1(C_1)}{\partial C_{12}} = \frac{\partial U_2(C_2)}{\partial C_{21}} \Big/ \frac{\partial U_2(C_2)}{\partial C_{22}} \quad . \qquad \text{(E3.11)}$$

Under (E3.2), (E3.11) implies that

$$\frac{C_{12}}{C_{11}} = \frac{C_{22}}{C_{21}} \quad . \qquad \text{(E3.12)}$$

To derive the utility possibilities frontier, we eliminate the four $C_{ij}$s from the five equations (E3.7) – (E3.10) and (E3.12) leaving us eventually with an equation of the form $f(U_1, U_2) = 0$. As the first step in the algebra we note that (E3.7) and (E3.8) imply that

$$U_1 U_2 = (C_{11})^{1/2} (C_{22})^{1/2} (C_{21})^{1/2} (C_{12})^{1/2} \quad . \qquad \text{(E3.13)}$$

On using (E3.12) in (E3.13), we obtain

$$U_1 U_2 = C_{11} C_{22} = C_{21} C_{12} \qquad \text{(E3.14)}$$

Next, we use (E3.9) and (E3.10) to find that

$$C_{11} C_{12} + C_{12} C_{21} + C_{11} C_{22} + C_{21} C_{22} = 10{,}000 \quad . \qquad \text{(E3.15)}$$

Substitution into (E3.15) from (E3.7), (E3.8) and (E3.14) gives

$$U_1^2 + 2U_1U_2 + U_2^2 = 10{,}000 \quad ,$$

that is

$$(U_1 + U_2)^2 - 10{,}000 = 0 \quad . \qquad \text{(E3.16)}$$

In view of (E3.7) and (E3.8) we can assume that $U_1$ and $U_2$ are non-negative. Thus, we may write the equation to the utility possibilities frontier as

$$U_1 + U_2 - 100 = 0 \quad . \qquad \text{(E3.17)}$$

The utility possibilities set is illustrated in Figure E3.1(a).

**(b)** We start by replacing (E3.7) and (E3.8) by

$$U_i^2 = (C_{i1})^{1/2}(C_{i2})^{1/2} \quad , \qquad i=1,2 \quad .$$

(E3.9), (E3.10) and (E3.12) remain valid. Thus, we can follow the steps in part (a), replacing $U_i$ wherever it appears with $U_i^2$. Consequently, it follows from (E3.17) that the equation for the utility possibilities frontier is

$$U_1^2 + U_2^2 - 100 = 0 \quad . \qquad \text{(E3.18)}$$

The utility possibilities set is illustrated in Figure E3.1(b).

**(c)** We replace (E3.7) by

$$U_1^2 = (C_{11})^{1/2}(C_{12})^{1/2} \quad .$$

(E3.8), (E3.9), (E3.10) and (E3.12) remain valid. It follows from part (a) that the equation for the utility possibilities frontier is

$$U_1^2 + U_2 - 100 = 0 \quad .$$

The utility possibilities set is illustrated in Figure E3.1(c).

**(d)** Equation (E3.6) can be rewritten as

$$\left[e^{U_i}\right]^{1/2} = (C_{i1})^{1/2}(C_{i2})^{1/2} \quad , \qquad i=1,\ 2 \quad .$$

(E3.9), (E3.10) and (E3.12) remain valid. Thus, from part (a) we can conclude that the equation to the utility possibilities frontier is

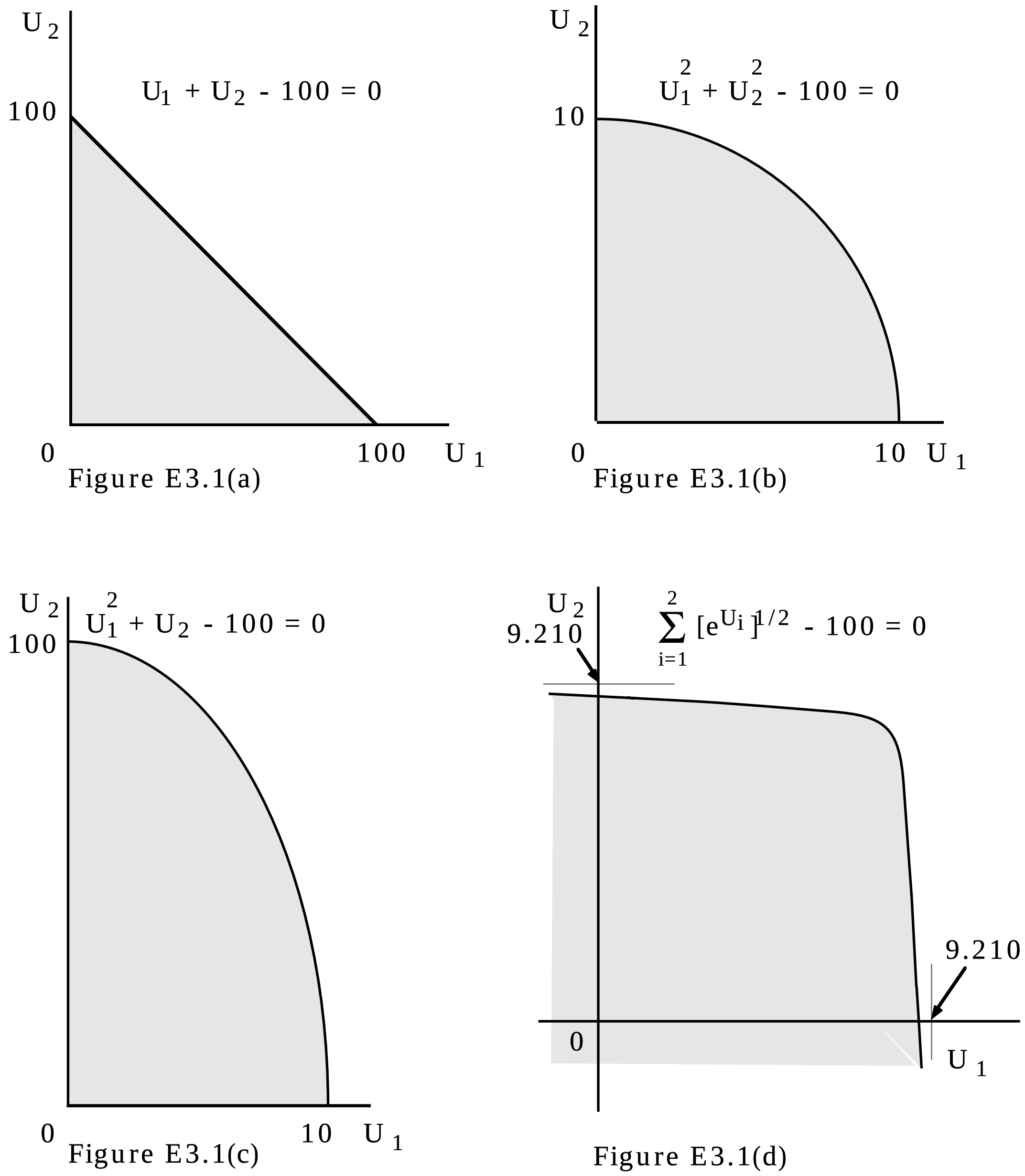


Figure E3.1 Utility possibilities frontiers

Utility possibilities sets are indicated by shading.

$$\sum_{i=1}^{2} [e^{U_i}]^{1/2} - 100 = 0 \quad .$$

The utility possibilities set is illustrated in Figure E3.1(d).

**(e)** First we introduce some notation. Let V(h) be a point in the utility possibilities set. Then we can write V(h) as

$$V(h) = \big(U_1(C_1(h)),\ U_2(C_2(h)),\ldots,U_r(C_r(h))\big)$$

where $C_1(h)$, $C_2(h),\ldots,C_r(h)$ are consumption vectors for consumers 1 to r underlying the point V(h).

Next we recall the definitions of a convex set and a concave function. Let V(1) and V(2) be any two points in the utility possibilities set. Then this set is convex if for any $\alpha \in [0, 1]$,

$$V(1,2,\alpha) \equiv \alpha\ V(1) + (1 - \alpha)\ V(2)$$

is in the utility possibilities set.

$U_i$ is a concave function over the convex set S (S might, for example, be the positive quadrant) if

$$U_i\big(\alpha\, C_i(1) + (1-\alpha)\ C_i(2)\big) \geq \alpha\ U_i\big(C_i(1)\big) + (1-\alpha)\ U_i\big(C_i(2)\big)$$

where $C_i(1)$ and $C_i(2)$ are any two points in S and $\alpha$ is any point in [0, 1].

Now we prove the proposition, i.e., we prove that if V(1) and V(2) are any two points in the utility possibilities set, then $V(1,2,\alpha)$ is also in the utility possibilities set for all $\alpha \in [0, 1]$. We start by noting that the two lists of consumption vectors $\big(C_1(h),\ldots,C_r(h)\big)$, h=1,2, underlying the utility vectors V(1) and V(2) satisfy

$$\sum_{i=1}^{r} C_i(h) \leq Z \quad , \qquad h=1,2 \quad , \tag{E3.19}$$

where Z is the combined endowment vector of the r consumers. Thus,

$$\sum_{i=1}^{r} C_i(1,2,\alpha) \;\le\; Z \qquad \text{(E3.20)}$$

where

$$C_i(1,2,\alpha) \;\equiv\; \alpha\, C_i(1) \;+\; (1-\alpha)\, C_i(2) \quad .$$

By concavity of the $U_i$, we have

$$V(1,2,\alpha) \;\le\; \big(U_1(C_1(1,2,\alpha)),\ldots,U_r(C_r(1,2,\alpha))\big)$$

$$\text{for all } \alpha \in [0,\ 1]\,. \qquad \text{(E3.21)}$$

In view of (E3.20), we know that the utility vector on the right hand side of (E3.21) is achievable. We may conclude that $V(1,2,\alpha)$ is also achievable, i.e., $V(1,2,\alpha)$ belongs to the utility possibilities set.

**(f)** Using the same notation as in part (e), we note that $(\sum_i C_i(1))$ and $(\sum_i C_i(2))$ are producible vectors, i.e., they lie in the net production possibilities set. By the convexity of this set, we know that $(\sum_i C_i(1,2,\alpha))$ is also a member for all $\alpha\in[0,\ 1]$. The concavity of the utility functions means that (E3.21) is still valid. Thus we can conclude that $V(1,2,\alpha)$ belongs to the utility possibilities set.

***Exercise 4 A multiple-consumer linear economy***

Consider an economy in which an equilibrium is a list of non-negative vectors and scalars $\{p,\ \gamma(1),\ \gamma(2),\ldots,\gamma(r),\ x\}$ satisfying

$$\sum_{k=1}^{r} a(k)\ \gamma(k) \;\le\; \sum_{k=1}^{r} z(k) + Ax \quad , \qquad \text{(E4.1)}$$

$$p'\left(\sum_{k=1}^{r} a(k)\ \gamma(k) - \sum_{k=1}^{r} z(k) - Ax\right) = 0 \quad , \qquad \text{(E4.2)}$$

$$p'A \;\le\; 0 \quad , \qquad \text{(E4.3)}$$

$$p'Ax \;=\; 0 \quad , \qquad \text{(E4.4)}$$

$$p'a(1) \;=\; 1 \qquad \text{(E4.5)}$$

and

$$p'z(k) = \gamma(k)\ p'a(k) \ , \quad k=1,\ldots,r-1 \ , \qquad \text{(E4.6)}$$

where p is the $n \times 1$ vector of commodity prices, $\gamma(k)$ is a scalar indicating the number of commodity bundles consumed by household k and x is an $m \times 1$ vector of production activity levels. The exogenous variables are a(k), k=1,...,r, the $n \times 1$ vector giving the commodity composition of the $k^{th}$ household's consumption bundle; z(k), k=1,...,r, the $n \times 1$ vector giving the $k^{th}$ household's resource endowment, and A, the $n \times m$ production technology matrix.

Conditions (E4.1) and (E4.2) impose market clearing. Conditions (E4.3) and (E4.4) impose zero profits. Condition (E4.5) sets the absolute price level. (The price level could, of course, be set in many other ways. No special significance should be attached to the way chosen here.) Condition (E4.6) provides the budget constraints for households 1,...,r–1. There is no need to include the budget constraint for household r. It is implied by the other conditions. (This follows easily from (E4.2), (E4.4) and (E4.6).)

(a) Assume that n = 3, m = 2 and r = 2, and that

$$a(1) = \begin{pmatrix} 5 \\ 4 \\ 0 \end{pmatrix}, \quad a(2) = \begin{pmatrix} 4 \\ 5 \\ 0 \end{pmatrix}, \quad z(1) = \begin{pmatrix} 0 \\ 0 \\ 25 \end{pmatrix}, \quad z(2) = \begin{pmatrix} 0 \\ 0 \\ 75 \end{pmatrix}$$

$$\text{and} \quad A = \begin{pmatrix} 1.0 & 0.0 \\ -0.2 & 1.0 \\ -1.0 & -1.0 \end{pmatrix} .$$

What are the equilibrium values for p, $\gamma(1)$, $\gamma(2)$ and x?

(b) The solution to part (a) is particularly easy because the price vector can be determined independently of demand. Samuelson's non-substitution theorem is applicable, see for example Baumol (1972, ch.20, section 4.1) or DBK pp. 264-267. When there is more than one non-producible commodity or factor, prices depend on the composition of consumer demand. The composition of demand depends on the values of the consumers' endowments which depend on prices. Thus, it is not possible in general to determine part of the solution (e.g., the prices) of model (E4.1) – (E4.6) independently of the rest of the solution. Suggest how (E4.1) – (E4.6) might be solved by a sequence of linear programs.

***Answer to Exercise 4***

**(a)** The equilibrium prices are the same as in Exercise 2 (a), i.e.,

$$(p_1, p_2, p_3) = (0.12, 0.10, 0.10) \quad .$$

This gives household 1 an income, $p'z(1)$, of (0.10)(25) = 2.50. The cost, $p'a(1)$, of a consumption bundle for this household is 1.00. Hence $\gamma(1) = 2.50$. The income of household 2 is (0.10)(75) = 7.50. The cost of 2's consumption bundle is (4)(0.12) + (5)(0.10) = 0.98. Hence $\gamma(2) = 7.50/0.98 = 7.65$. We can now determine x from

$$2.50 \begin{pmatrix} 5 \\ 4 \end{pmatrix} + 7.65 \begin{pmatrix} 4 \\ 5 \end{pmatrix} = \begin{pmatrix} 1.0 & 0.0 \\ -0.2 & 1.0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} ,$$

This gives $(x_1, x_2) = (43.1 \quad 56.9)$ .

**(b)** One approach to computing solutions for the model (E4.1) – (E4.6) is to solve a sequence of linear programming problems in which the consumption level for one household is maximized subject to each of the other households achieving given consumption levels. If we choose to maximize consumption for the first household, then the $s^{th}$ linear program in the sequence has the form: find non-negative values for $\gamma(1)$ and x to

$$\left.\begin{array}{l} \text{maximize } \gamma(1) \\ \text{subject to} \\ \gamma(1)a(1) \le X^{(s)} + Ax, \end{array}\right\} \qquad \text{(E4.7)}$$

where the n × 1 vector $X^{(s)}$ is an iterative variable, i.e., it is varied as we move from one linear programming problem to the next but it is held constant within each linear programming problem.

Initially, we set X according to

$$X^{(1)} = \sum_{k=1}^{r} z(k) - \sum_{k=2}^{r} a(k)\ \gamma^{(1)}(k)$$

where the $\gamma^{(1)}(k)$, k=2, ..., r are guesses of the equilibrium values of $\gamma(k)$, k=2, ..., r. One possibility is to set

$$\gamma^{(1)}(k) = 0 \quad , \quad k=2, \ldots, r \quad .$$

This may not be a realistic guess. However, it does ensure that problem (E4.7) is feasible. If the $\gamma^{(1)}(k)$s were set too high, it is possible that there would be no non-negative values for $\gamma(1)$ and x which would satisfy the constraints in (E4.7).

Having selected values for the $\gamma^{(1)}(k)$, k=2, ..., r, we compute $X^{(1)}$ and solve the problem (E4.7) for s = 1.

If $\gamma^{(1)}(1)$ and $x^{(1)}$ are a solution to this problem, then there exists $p^{(1)} \geq 0$ such that

$$1 - \left(p^{(1)}\right)' a(1) = 0 \quad ,$$

$$\left(p^{(1)}\right)' A \leq 0 \quad ,$$

$$\left(p^{(1)}\right)' A x^{(1)} = 0 \quad ,$$

$$\gamma^{(1)}(1)\ a(1) \leq X^{(1)} + A x^{(1)}$$

and

$$\left(p^{(1)}\right)' \left(\gamma^{(1)}(1)\ a(1) - X^{(1)} - A x^{(1)}\right) = 0 \quad .$$

If in addition it happens that $p^{(1)}$ and $\gamma^{(1)}(1)$, together with the guesses $\gamma^{(1)}(k)$, k=2, ..., r, satisfy

$$\left(p^{(1)}\right)' z(k) = \gamma^{(1)}(k) \left(p^{(1)}\right)' a(k) \quad , \quad k=1, \ldots, r-1 \quad , \qquad \text{(E4.8)}$$

then it is clear that

$$\left\{ p^{(1)}, \gamma^{(1)}(1), \ldots, \gamma^{(1)}(r), x^{(1)} \right\}$$

is a solution to the model (E4.1) – (E4.6).

Normally, we will not be so lucky that (E4.8) is satisfied. We proceed by updating our guesses for $\gamma(k)$, k=2, ..., r, recomputing X and re-solving (E4.7). An updating formula which is often satisfactory is

$$\gamma^{(s)}(k) \;=\; \left(p^{(s-1)}\right)' z(k) \,\Big/\, \left(p^{(s-1)}\right)' a(k) \;, \quad k=2, \ldots, r, \qquad \text{(E4.9)}$$

i.e., our guess for $\gamma(k)$, k=2, ..., r, to be used in the $s^{th}$ linear program is obtained by calculating the number of consumption bundles which consumer k could afford at the commodity prices, $p^{(s-1)}$, revealed from the solution to our $(s-1)^{th}$ linear program.

When (E4.9) gives

$$\gamma^{s}(k) \;=\; \gamma^{s-1}(k) \;, \quad k=2, \ldots, r \;,$$

then convergence has occurred and we have found a solution to the model (E4.1) – (E4.6). Convergence cannot be guaranteed. In practice, however, convergence is usually achieved with a small value for s (e.g., s = 3). The reason is that the value for p emerging from (E4.7) is normally fairly insensitive to variations in X. If there is only one non-producible commodity, (the Samuelson non-substitution situation) then p is completely insensitive to variations in X. In this case, the search for an equilibrium for the model (E4.1) – (E4.6) will necessitate the solution of no more than two linear programming problems of the form (E4.7).

***Exercise 5 A linear economy with a utility maximizing consumer***

Consider an economy in which an equilibrium is a list of non-negative vectors and scalars {p, c, Y, x} satisfying:

$$\left.\begin{array}{l} c \text{ maximizes } U(c) \\ \text{subject to} \\ p'c = Y \end{array}\right\} \qquad \text{(E5.1)}$$

$$\left.\begin{array}{l} c \le z + Ax \;, \\ p'(c - z - Ax) = 0, \end{array}\right\} \qquad \text{(E5.2)}$$

$$\left.\begin{array}{l} p'A \le 0 \;, \\ p'Ax = 0 \end{array}\right\} \qquad \text{(E5.3)}$$

and

$$p'z = 1 \quad , \tag{E5.4}$$

where p is the n × 1 vector of commodity prices, c is the n × 1 vector of household consumption by commodity, the scalar Y is household expenditure and x is the m × 1 vector of production activity levels. The exogenous variables are z, the n × 1 vector giving the economy's resource endowments and A, the n × m production technology matrix. U is a utility function describing household preferences.

Condition (E5.1) says that the consumption vector, c, is chosen to maximize utility subject to the household budget constraint. Conditions (E5.2) and (E5.3) are familiar from earlier exercises. Condition (E5.4) sets the absolute price level.[15]

(a) For (p, c, Y, x) satisfying (E5.1) – (E5.4), show that

$$Y = p'z \quad . \tag{E5.5}$$

(b) Assume that

$$U(c) = \max_{\gamma} \{\gamma \mid a\gamma \le c\} \tag{E5.6}$$

where a is a non-negative n × 1 vector. Assume in addition that n = 3 and

$$a = \begin{pmatrix} 5 \\ 4 \\ 0 \end{pmatrix} \quad . \tag{E5.7}$$

Evaluate

$$U\begin{pmatrix} 5 \\ 4 \\ 0 \end{pmatrix} , \quad U\begin{pmatrix} 6 \\ 4 \\ 0 \end{pmatrix} , \quad U\begin{pmatrix} 5 \\ 5 \\ 5 \end{pmatrix} , \quad U\begin{pmatrix} 5 \\ 5 \\ 0 \end{pmatrix} \quad .$$

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15 In previous exercises we fixed the absolute value of a consumption bundle, see (E2.3). Here, where the composition of the consumption bundle is potentially endogenous, it is convenient to fix, instead, the absolute value of the resource endowment.

Sketch some indifference curves for the utility function (E5.6) – (E5.7) in commodity 1/commodity 2 space. Solve the model (E5.1) – (E5.4) assuming that

$$z = \begin{pmatrix} 0 \\ 0 \\ 100 \end{pmatrix} , \qquad A = \begin{pmatrix} 1.0 & 0.0 \\ -0.2 & 1.0 \\ -1.0 & -1.0 \end{pmatrix} , \qquad \text{(E5.8)}$$

and that U(c) is given by (E5.6) – (E5.7).

(c) Assume that

$$U(c) = \max_{\gamma} \{1_h' \gamma \mid a\gamma \le c, \gamma \ge 0\} \text{ for } c \ge 0 , \qquad \text{(E5.9)}$$

where a is now a non-negative n × h matrix and γ is an h × 1 vector. $1_h'$ is a 1 × h vector of ones. Assume in addition that n = 3, h = 2 and that

$$a = \begin{pmatrix} 3 & 2 \\ 2 & 3 \\ 0 & 0 \end{pmatrix} . \qquad \text{(E5.10)}$$

Evaluate

$$U\begin{pmatrix} 30 \\ 20 \\ 0 \end{pmatrix} , U\begin{pmatrix} 20 \\ 30 \\ 0 \end{pmatrix} , U\begin{pmatrix} 25 \\ 25 \\ 0 \end{pmatrix} , U\begin{pmatrix} 40 \\ 20 \\ 0 \end{pmatrix} , U\begin{pmatrix} 20 \\ 40 \\ 0 \end{pmatrix} .$$

Sketch some indifference curves for utility function (E5.9) – (E5.10) in commodity 1/commodity 2 space. Solve the model (E5.1) – (E5.4) assuming that z and A are given by (E5.8) and that U(c) is given by (E5.9) – (E5.10).

(d) Write a linear programming problem which would be a suitable vehicle for solving the model (E5.1) – (E5.4) where the utility function takes the form (E5.9).

(e) In a utility maximizing model with the utility specification (E5.9), the expenditure elasticities of demand for all goods are unity. A specification which allows expenditure elasticities to vary from unity but is still convenient for use in a linear programming framework is

$$U(c) = \max_{\gamma} \{1_h' \gamma \mid a\gamma \le c - b;\ \gamma \ge 0\} \quad \text{for } c \ge b, \tag{E5.11}$$

where a is a non-negative n × h matrix and b is an n × 1 vector (components of b may have either sign). In using (E5.11), we assume that b is sufficiently small that the possibility that $c \ngeq b$ can be ignored.

Assume that

$$b = \begin{pmatrix} 10 \\ -10 \\ 0 \end{pmatrix} \tag{E5.12}$$

and that a is given by (E5.10). Answer the questions in part (c) under this new utility specification.

(f) Write a linear programming problem which would be a suitable vehicle for solving the model (E5.1) – (E5.4) where the utility function takes the form (E5.11).

***Answer to Exercise 5***

**(a)** From (E5.2) and (E5.3) we have

$$p'c - p'z = 0$$

and from (E5.1) we have

$$p'c = Y \quad .$$

Thus,

$$Y = p'z \quad ,$$

i.e., household expenditure equals the value of the economy's resource endowment.

**(b)**

$$U\begin{pmatrix} 5 \\ 4 \\ 0 \end{pmatrix} = \max_{\gamma} \left\{ \gamma \,\middle|\, \begin{pmatrix} 5 \\ 4 \\ 0 \end{pmatrix} \gamma \le \begin{pmatrix} 5 \\ 4 \\ 0 \end{pmatrix} \right\} .$$

The largest value for γ which is consistent with

$$\begin{pmatrix} 5 \\ 4 \\ 0 \end{pmatrix} \gamma \le \begin{pmatrix} 5 \\ 4 \\ 0 \end{pmatrix}$$

is $\gamma = 1$. Thus $U\begin{pmatrix}5\\4\\0\end{pmatrix} = 1$. Similarly, it may be shown that

$$U\begin{pmatrix}6\\4\\0\end{pmatrix} = U\begin{pmatrix}5\\5\\5\end{pmatrix} = U\begin{pmatrix}5\\5\\0\end{pmatrix} = 1 \quad .$$

Part of the indifference map for utility function (E5.6) – (E5.7) is in Figure E5.1.

The solution to the model (E5.1) – (E5.4) with A and z given by (E5.8) and U given by (E5.6) – (E5.7) is

$$p' = (0.012, \; 0.010, \; 0.010) \quad ,$$

$$c' = (50, 40, 0) \quad ,$$

$$Y = 1$$

and

$$x' = (50, 50) \quad .$$

Apart from the normalization of prices, the model is the same as that in Exercise 2, part (a). Notice that a household with utility function (E5.6) – (E5.7) will always consume goods 1 and 2 in the ratio 5 to 4.

**(c)** $$U\begin{pmatrix}30\\20\\0\end{pmatrix} = \max_{\gamma_1,\gamma_2} \left\{ \gamma_1 + \gamma_2 \;\middle|\; \begin{pmatrix}3 & 2\\2 & 3\\0 & 0\end{pmatrix}\begin{pmatrix}\gamma_1\\\gamma_2\end{pmatrix} \le \begin{pmatrix}30\\20\\0\end{pmatrix}; \gamma_1,\gamma_2 \ge 0 \right\} .$$

The largest value for $\gamma_1 + \gamma_2$ which is consistent with

$$\begin{pmatrix}3 & 2\\2 & 3\\0 & 0\end{pmatrix}\begin{pmatrix}\gamma_1\\\gamma_2\end{pmatrix} \le \begin{pmatrix}30\\20\\0\end{pmatrix} ; \qquad \gamma_1,\gamma_2 \ge 0 ,$$

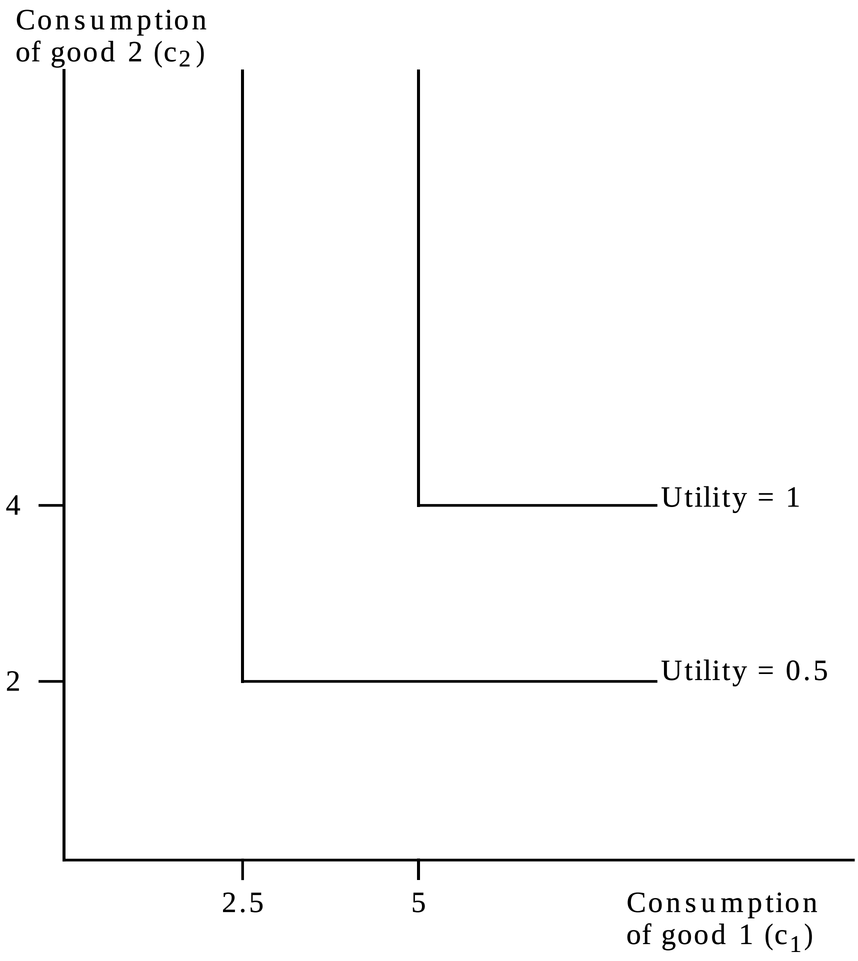


Figure E5.1 Indifference curves for a consumer having utility function (E5.6) - (E5.7)

occurs when $\gamma_1 = 10$ and $\gamma_2 = 0$. Hence,

$$U\begin{pmatrix}30\\20\\0\end{pmatrix} = 10 \quad .$$

Similarly, it may be shown that

$$U\begin{pmatrix}20\\30\\0\end{pmatrix} = U\begin{pmatrix}25\\25\\0\end{pmatrix} = U\begin{pmatrix}40\\20\\0\end{pmatrix} = U\begin{pmatrix}20\\40\\0\end{pmatrix} = 10 \quad .$$

Part of the indifference map for utility function (E5.9) – (E5.10) is in Figure E5.2.

With only one non-producible good, a change in the utility function does not affect the equilibrium prices in the model (E5.1) – (E5.4). Thus, as in part (a), we have

$$p' = (0.012,\ 0.010,\ 0.010)$$

and

$$Y = 1 \quad . \tag{E5.13}$$

The household budget constraint is

$$0.012c_1 + 0.010c_2 + 0.010c_3 = 1 \quad . \tag{E5.14}$$

Under (E5.9) – (E5.10), consumption of good 3 does not contribute to utility. Thus $c_3 = 0$ and (E5.14) may be reduced to

$$0.012c_1 + 0.010c_2 = 1 \quad . \tag{E5.15}$$

By representing (E5.15) in Figure E5.2 we find that the utility maximizing consumption vector consistent with the budget constraint is

$$c' = (37.0,\ 55.6,\ 0) \quad .$$

Finally, we compute the activity levels, x, from

$$\begin{pmatrix} 37.0 \\ 55.6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1.0 & 0.0 \\ -0.2 & 1.0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad , \tag{E5.16}$$

giving

$$x' = (37.0,\ 63.0) \quad .$$

In (E5.16), the market clearing equation for commodity 3 has been omitted. On the other hand, in (E5.13) we have used (E5.5). With the value of consumption equal to the value of resources, demand equal to supply for goods 1 and 2, zero pure profits in all production activities, and $p_3 > 0$, it is easy to show that demand must equal supply for commodity 3.

**(d)** Consider the problem of choosing non-negative values for $\gamma$ and x to

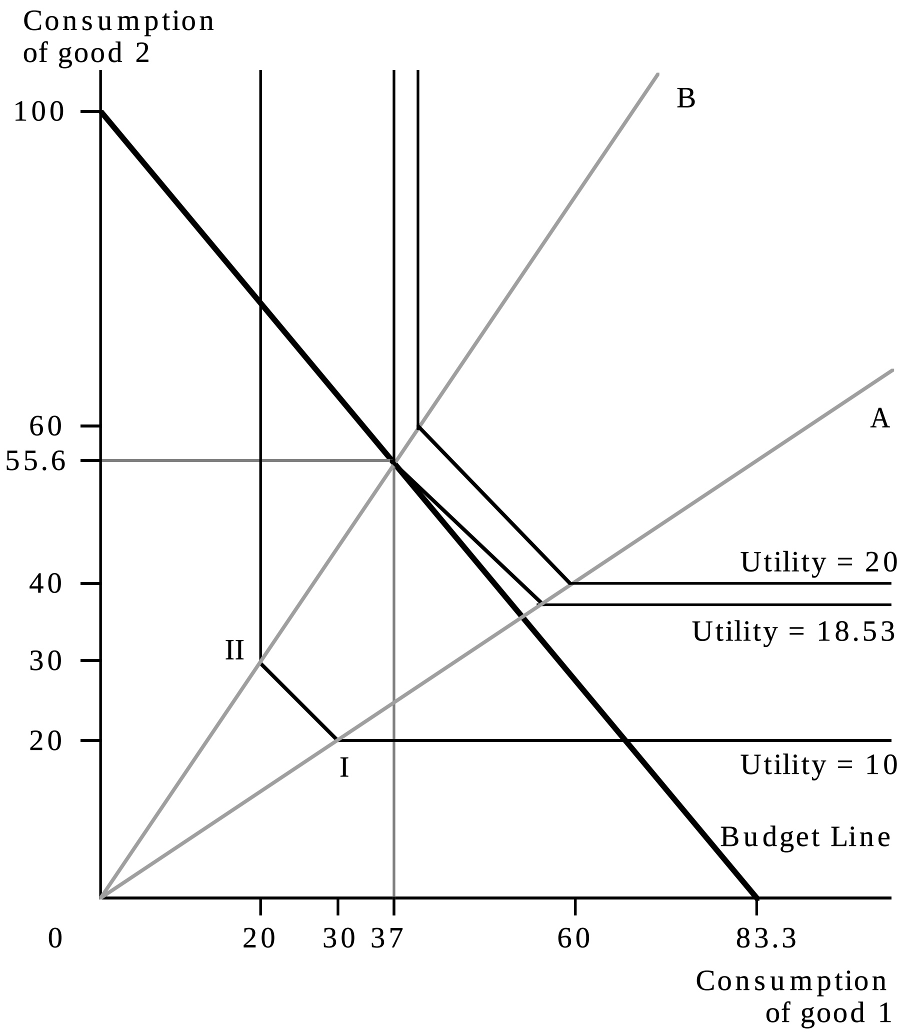


Figure E5.2 Indifference map and budget line for the consumer in model (E5.1) - (E5.4), assuming (E5.8) - (E5.10)

Indifference curves may be constructed by using the two rays 0A and 0B. With the consumption bundle at point I on ray 0A, we have utility of 10($\gamma_1$ = 10, $\gamma_2$ = 0). At point II on ray 0B, utility is also 10($\gamma_1$ = 0, $\gamma_2$ = 10). Points on the straight line between I and II give utility of 10 with both consumption activities being used. For example, the point half way between I and II (i.e., $c_1$ = 25, $c_2$ = 25) gives utility of 10 with $\gamma_1$ = 5 and $\gamma_2$ = 5.

$$\text{maximize} \quad 1_h' \gamma \tag{E5.17}$$

$$\text{subject to} \quad a\gamma \le z + Ax \quad . \tag{E5.18}$$

The non-negative vectors $\bar{\gamma}$ and $\bar{x}$ are a solution to this problem if and only if there exists $\bar{p} \geq 0$, such that

$$1_h' - \bar{p}'a \le 0 \quad , \tag{E5.19}$$

$$(1_h' - \bar{p}'a)\bar{\gamma} = 0 \quad , \tag{E5.20}$$

$$\bar{p}'A \le 0 \quad , \tag{E5.21}$$

$$\bar{p}'A\bar{x} = 0 \quad , \tag{E5.22}$$

$$a\bar{\gamma} - z - A\bar{x} \le 0 \tag{E5.23}$$

and

$$\bar{p}'(a\bar{\gamma} - z - A\bar{x}) = 0 \quad . \tag{E5.24}$$

Now we show that

$$p = \left(\frac{1}{\bar{p}'z}\right) \bar{p} \quad , \qquad \text{(E5.25)}[16]$$

$$c = a\bar{\gamma} \quad , \tag{E5.26}$$

$$Y = \left(\frac{1}{\bar{p}'z}\right) \bar{p}'a\bar{\gamma} \quad , \tag{E5.27}$$

and

$$x = \bar{x} \tag{E5.28}$$

is a solution to the model (E5.1) – (E5.4) where the utility function has the form (E5.9). When we have done this, we will have established that the model can be solved by first finding the primal and dual solutions to the linear programming problem (E5.17) – (E5.18) and then using formulae (E5.25) – (E5.28).

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16 In writing (E5.25), we assume that $\bar{p}'z > 0$.

Clearly our suggested solution (E5.25) – (E5.28) satisfies conditions (E5.2) – (E5.4). Does it satisfy (E5.1)? Under (E5.9), the solution for the consumer problem (E5.1) is the solution for c in the problem of choosing non-negative values for c and $\gamma$ to

$$\text{maximize} \quad 1_h' \gamma \tag{E5.29}$$

$$\text{subject to} \quad a\gamma \leq c \tag{E5.30}$$

$$\text{and} \quad p'c \leq Y \; . \tag{E5.31}$$

Non-negative vectors $\gamma$ and c are a solution of this problem if and only if there exist $q \geq 0$ and $\lambda \geq 0$ such that

$$\left.\begin{array}{l} 1_h' - q'a \leq 0, \\ (1_h' - q'a)\gamma = 0, \end{array}\right\} \tag{E5.32}$$

$$\left.\begin{array}{l} q' - \lambda p' \leq 0, \\ (q' - \lambda p')c = 0 \; , \end{array}\right\} \tag{E5.33}$$

$$\left.\begin{array}{l} a\gamma - c \leq 0, \\ q'(a\gamma - c) = 0, \end{array}\right\} \tag{E5.34}$$

$$\left.\begin{array}{l} p'c - Y \leq 0 \\ \text{and} \\ \lambda(p'c - Y) = 0. \end{array}\right\} \tag{E5.35}$$

If $\bar{p}$ and $\bar{Y}$ are given by (E5.25) and (E5.27), then it is apparent that $c = a\gamma$ and $\gamma = \bar{\gamma}$ is a solution to (E5.29) – (E5.31). (Conditions (E5.32) – (E5.35) are satisfied with $\lambda = \bar{p}'z$ and $q = \bar{p}$.) We may conclude that our suggested solution (E5.25) – (E5.28) for the model (E5.1) – (E5.4) is consistent with the utility maximizing condition (E5.1) where the utility function is of the form (E5.9).

**(e)**

$$U\begin{pmatrix}30\\20\\0\end{pmatrix} = \max_{\gamma_1,\gamma_2}\left\{\gamma_1 + \gamma_2 \;\middle|\; \begin{pmatrix}3 & 2\\2 & 3\\0 & 0\end{pmatrix}\begin{pmatrix}\gamma_1\\ \gamma_2\end{pmatrix} \le \begin{pmatrix}30\\20\\0\end{pmatrix} - \begin{pmatrix}10\\-10\\0\end{pmatrix};\ \gamma_1,\gamma_2 \ge 0\right\} \quad \text{(E5.36)}$$

Figure E5.3 shows that the solution to the problem on the right hand side of (E5.36) is $\gamma_1 = 0$, $\gamma_2 = 10$. Hence $U\begin{pmatrix}30\\20\\0\end{pmatrix} = 10$. Similarly, it may be shown that

$$U\begin{pmatrix}20\\30\\0\end{pmatrix} = 5\ , \qquad U\begin{pmatrix}25\\25\\0\end{pmatrix} = 7\tfrac{1}{2}\ , \qquad U\begin{pmatrix}40\\20\\0\end{pmatrix} = 12 \quad \text{and} \quad U\begin{pmatrix}20\\40\\0\end{pmatrix} = 5 \quad .$$

Part of the indifference map for the utility function (E5.11) with parameter values given by (E5.10) and (E5.12) is in Figure E5.4.

With A and z given by (E5.8), the equilibrium prices and expenditure level continue to be as in parts (b) and (c), that is

$$p' = (0.012,\ 0.010,\ 0.010)$$

and

$$Y = 1 \quad .$$

The household budget constraint is still given by (E5.14) which reduces to (E5.15) when it is recognized that good 3 is not consumed. By drawing the budget constraint in Figure E5.4, we find that

$$c' = (46.3,\ 44.4,\ 0) \quad .$$

Finally we compute the activity levels, x, from

$$\begin{pmatrix}46.3\\44.4\end{pmatrix} = \begin{pmatrix}1.0 & 0.0\\-0.2 & 1.0\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix} \ ,$$

giving

$$x' = (46.3,\ 53.7) \quad .$$

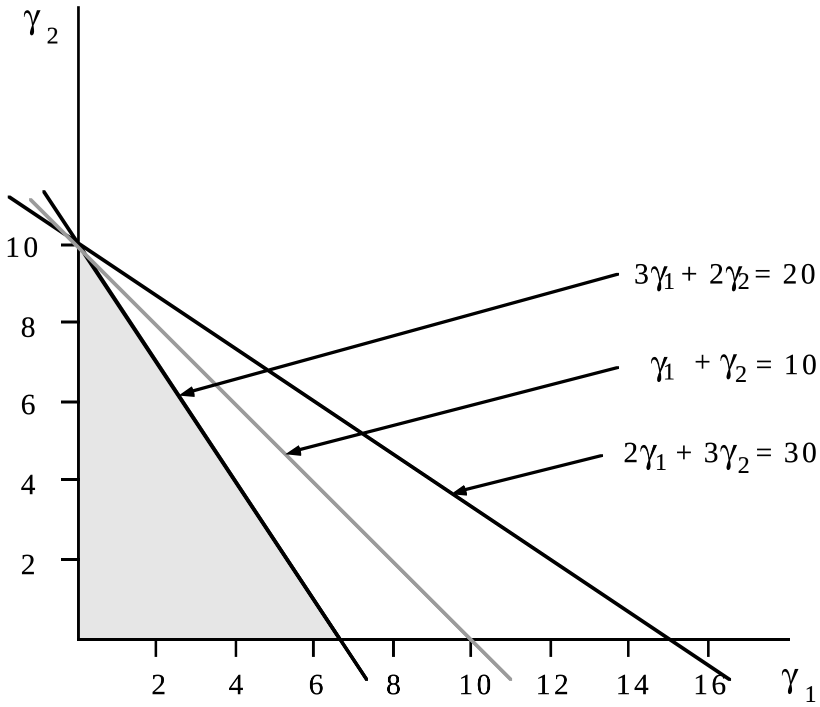


Figure E5.3 Graphical solution to the problem on the right hand side of (E5.36).

The shading indicates the values of $\gamma_1$ and $\gamma_2$ which are consistent with the constraints.

**(f)** An appropriate linear programming problem is: choose non-negative values for $\gamma$ and x to

maximize

$$1_h' \gamma \tag{E5.37}$$

subject to

$$a\gamma + b \leq z + Ax \quad . \tag{E5.38}$$

Where $\bar{\gamma}$, $\bar{x}$ and $\bar{p}$ are a primal and associated dual solution to this problem, it can be shown, by following the method in part (d), that

$$p = \left(\frac{1}{\bar{p}'z}\right)\bar{p} \quad ,$$

$$c = a\bar{\gamma} + b \quad ,$$

$$Y = \left(\frac{1}{\bar{p}'z}\right)\left(\bar{p}'a\bar{\gamma} + \bar{p}'b\right) \quad ,$$

and

$$x = \bar{x}$$

is a solution to the model (E5.1) – (E5.4) when the utility function has the form (E5.11).

## *Exercise 6 A linear economy with several utility maximizing consumers*

Consider an economy in which an equilibrium is a list of non-negative vectors and scalars

{p, c(1), c(2), …, c(r), Y(1), …, Y(r), x} satisfying the following conditions:

$$\left.\begin{array}{l}\text{for each } k, k=1,\dots,r, \ c(k) \text{ maximizes} \\ U_k\big(c(k)\big) \text{ subject to } p'c(k)=Y(k),\end{array}\right\} \qquad (E6.1)$$

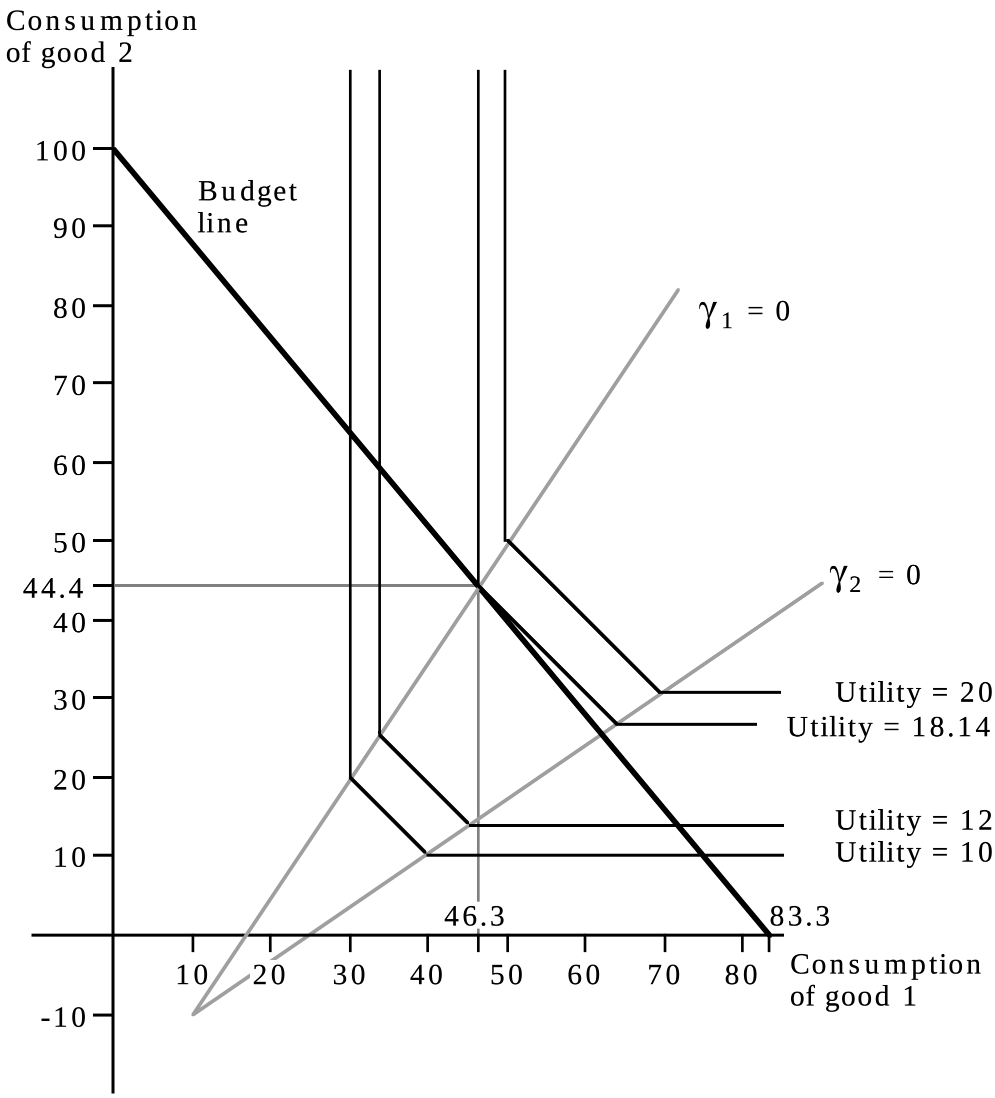


Figure E5.4 Indifference map and budget line for the consumer in model (E5.1) - (E5.4), assuming (E5.8) and (E5.10) - (E5.12)

Rather than using rays from the origin in the construction of the indifference curves, in this figure we use rays from the point b'= (10,-10). Experiment with some parallel shifts in the budget line to convince yourself that a consumer with the indifference map shown here would have expenditure elasticities of more than one for good 2 and less than one for good 1.

$$\sum_{k=1}^{r} c(k) \leq \sum_{k=1}^{r} z(k) + Ax \quad , \qquad (E6.2)$$

$$p'\left(\sum_{k=1}^{r} c(k) - \sum_{k=1}^{r} z(k) - Ax\right) = 0 \quad , \qquad (E6.3)$$

$$p'A \leq 0 \quad , \qquad (E6.4)$$

$$p'Ax = 0 \quad , \qquad (E6.5)$$

$$p'z(1) = 1 \qquad (E6.6)$$

and

$$p'z(k) = Y(k), \quad k=1, \ldots, r-1 \quad , \qquad (E6.7)$$

where p is the n × 1 vector of commodity prices, c(k) is the n × 1 consumption vector for household k, Y(k) is a scalar giving household k's income and x is the m × 1 vector of production activity levels. The exogenous variables are z(k) for k=1, ..., r, the n × 1 vectors giving the households' resource endowments, and A, the n × m production technology matrix. $U_k$ is a utility function describing household k's preferences.

(a) For $(\bar{p}, \bar{c}(1), \ldots, \bar{c}(r), \bar{Y}(1), \ldots, \bar{Y}(r), \bar{x})$ satisfying (E6.1) – (E6.7) show that

$$\bar{Y}(r) = \bar{p}'\bar{z}(r) \quad . \qquad (E6.8)$$

(b) Assume that

$$U_k(c(k)) = \max_{\gamma(k)} \{1_h' \gamma(k) \mid a(k)\gamma(k) \leq c(k) \ , \ \gamma(k) \geq 0\} \quad ,$$
$$k=1, \ldots, r \quad . \qquad (E6.9)$$

Find a solution for model (E6.1) – (E6.7) in the special case where there are two households with

$$a(1) = \begin{pmatrix} 3 & 1 \\ 2 & 3 \\ 0 & 0 \end{pmatrix} \quad , \quad a(2) = \begin{pmatrix} 4 & 5 \\ 4 & 3 \\ 0 & 0 \end{pmatrix} \quad , \qquad (E6.10)$$

$$z(1) = \begin{pmatrix} 0 \\ 0 \\ 50 \end{pmatrix} \quad \text{and} \quad z(2) = \begin{pmatrix} 0 \\ 0 \\ 150 \end{pmatrix} . \tag{E6.11}$$

Assume that the technology matrix is

$$A = \begin{pmatrix} 1.0 & -0.2 \\ -0.4 & 1.0 \\ -1.0 & -2.0 \end{pmatrix} . \tag{E6.12}$$

(c) Assuming that the utility functions have the form (E6.9), suggest how (E6.1) – (E6.7) might be solved by a sequence of linear programs.

***Answer to Exercise 6***

**(a)** From (E6.3) and (E6.5), we have

$$\bar{p}' \sum_{k=1}^{r} \bar{c}(k) = \bar{p}' \sum_{k=1}^{r} z(k) \quad ,$$

that is,

$$\sum_{k=1}^{r-1} \bar{p}' \bar{c}(k) + \bar{p}' c(r) = \sum_{k=1}^{r-1} \bar{p}' z(k) + \bar{p}' z(r) \quad . \tag{E6.13}$$

By using (E6.7) and the budget constraints from (E6.1), we find that (E6.13) reduces to (E6.8).

**(b)** There are no initial stocks of either goods 1 or 2 and both these goods are required if households are to have non–zero utility levels. We may assume, therefore, that both goods will be produced. This implies that both production activities are operated at positive levels. Consequently, we may determine the commodity prices from

$$(p_1, p_2, p_3) \begin{pmatrix} 1.0 & -0.2 & 0 \\ -0.4 & 1.0 & 0 \\ -1.0 & -2.0 & 50 \end{pmatrix} = (0, 0, 1) \quad .$$

We obtain

$$(p_1, p_2, p_3) = (0.0391, 0.0478, 0.0200) \quad . \tag{E6.14}$$

We also note that

$$Y(1) = 1.0 \text{ and } Y(2) = 3.0 \quad . \qquad \text{(E6.15)}$$

With an income of 1.0 and the prices given in (E6.14) household 1 will choose its consumption vector $c(1) \equiv (c_1(1), c_2(1))'$ and vector of consumption activity levels $\gamma(1) \equiv (\gamma_1(1), \gamma_2(1))'$ to

$$\text{maximize} \quad \gamma_1(1) + \gamma_2(1) \qquad \text{(E6.16)}$$

subject to

$$3\gamma_1(1) + \gamma_2(1) - c_1(1) \leq 0 \quad , \qquad \text{(E6.17)}$$

$$2\gamma_1(1) + 3\gamma_2(1) - c_2(1) \leq 0 \qquad \text{(E6.18)}$$

and

$$0.0391c_1(1) + 0.0478c_2(1) \leq 1 \quad . \qquad \text{(E6.19)}$$

At a solution to this problem, it is clear that constraints (E6.17) and (E6.18) will hold as equalities[17]. Thus we can eliminate $c_1(1)$ and $c_2(1)$ and consider the reduced problem of choosing non-negative values for $\gamma_1(1)$ and $\gamma_2(1)$ to

$$\text{maximize} \quad \gamma_1(1) + \gamma_2(1)$$

subject to

$$0.2129\gamma_1(1) + 0.1825\gamma_2(1) \leq 1 \quad .$$

The solution to this problem is

$$\gamma_1(1) = 0 \quad , \quad \gamma_2(1) = 5.48 \quad ,$$

giving

$$c_1(1) = 5.48 \quad , \quad c_2(1) = 16.44 \quad .$$

Similarly, you will find that

$$c_1(2) = 44.26 \quad , \quad c_2(2) = 26.56 \quad .$$

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17 Imagine that one of these constraints, say (E6.17), was binding while the other, (E6.18), was slack. Then we could increase $\gamma_1(1) + \gamma_2(1)$ by increasing $c_1(1)$ and decreasing $c_2(1)$. This could be done without violating (E6.19) by increasing $c_1(1)$ by 0.0478/0.0391 per unit decrease in $c_2(1)$.

Finally, we determine the production activity levels from any two equations in the system[18]

$$\begin{pmatrix} 49.74 \\ 43.00 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 200 \end{pmatrix} + \begin{pmatrix} 1.0 & -0.2 \\ -0.4 & 1.0 \\ -1.0 & -2.0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad . \tag{E6.20}$$

(E6.20) implies that

$$x_1 = 63.4 \text{ and } x_2 = 68.3 \quad .$$

**(c)** One method of computing solutions to the model (E6.1) – (E6.7) with the utility function given by (E6.9) is to solve a sequence of linear programs in which the $s^{th}$ program has the form: find non-negative values for $\gamma(k)$, k=1, ..., r and x to

$$\text{maximize} \qquad 1_h' \gamma(1) \tag{E6.21}$$

subject to

$$1_h' \gamma(k) \geq H^{(s)}(k) \quad , \quad k=2, \ldots, r \tag{E6.22}$$

and

$$\sum_{k=1}^{r} a(k)\,\gamma(k) - Ax \leq \sum_{k=1}^{r} z(k) \quad , \tag{E6.23}$$

where the $H^{(s)}(k)$, k=2, ..., r are k – 1 iterative variables which are varied as we move from one programming problem to the next but are held constant within each problem. This method is similar to that described in Exercise 4(b). We maximize utility for household 1 subject to each of the other households achieving given utility targets.

The first step in justifying the method and in suggesting how the iterative variables should be set is to write out the conditions for a solution of (E6.21) – (E6.23). If $\gamma^{(s)}(k)$, k=1, ..., r and $x^{(s)}$ are a solution to this problem, then there exist $p^{(s)} \geq 0$ and $\delta^{(s)}(k) \geq 0$, k=1, ..., r such that

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18 One equation in this sytem is redundant, see the discussion following (E5.16).

$$1_h' - \delta^{(s)}(k)\left(p^{(s)}\right)' a(k) \leq 0 \quad , \quad k=1, \ldots, r \quad , \tag{E6.24}$$

$$\left(1_h' - \delta^{(s)}(k)\left(p^{(s)}\right)' a(k)\right) \gamma^{(s)}(k) = 0 \quad , \quad k=1, \ldots, r \quad , \tag{E6.25}$$

$$\left(p^{(s)}\right)' A \leq 0 \quad , \tag{E6.26}$$

$$\left(p^{(s)}\right)' A\, x^{(s)} = 0 \quad , \tag{E6.27}$$

$$1_h' \gamma^{(s)}(k) = H^{(s)}(k) \quad , \quad k=2, \ldots, r \quad , \tag{E6.28}$$

$$\sum_{k=1}^{r} a(k)\, \gamma^{(s)}(k) - A\, x^{(s)} - \sum_{k=1}^{r} z(k) \leq 0 \tag{E6.29}$$

and

$$\left(p^{(s)}\right)' \left( \sum_{k=1}^{r} a(k)\, \gamma^{(s)}(k) - A x^{(s)} - \sum_{k=1}^{r} z(k) \right) = 0 \quad , \tag{E6.30}$$

where $\delta^{(s)}(1) = 1$. The $\delta^{(s)}(k)$, k=2, ..., r are the reciprocals of the Lagrangian multipliers associated with the constraints (E6.22). We assume that these constraints are binding for k=2, ..., r, and that $\delta^{(s)}(k) > 0$, k=2, ..., r. We could proceed without these assumptions. However, the subsequent discussion would become unnecessarily cumbersome. In terms of problem (E6.21) – (E6.23) we are assuming that a reduction in any of the utility targets, $H^{(s)}(k)$, k=2, ..., r, allows an increase in the optimal value of the objective function (E6.21).

Next we show that if

$$\left(p^{(s)}\right)' \left(a(k)\, \gamma^{(s)}(k) - z(k)\right) = 0 \quad , \quad k=1, \ldots, r \quad , \tag{E6.31}$$ [19]

then

$$p = \frac{1}{\left(p^{(s)}\right)' z(1)}\, p^{(s)} \quad , \tag{E6.32}$$

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19 We could restrict the subscript range in (E6.31) to k=2, ..., r. In view of (E6.27) and (E6.30), it is clear that if (E6.31) is valid for k=2, ..., r, then it is valid for k=1.

$$c(k) = a(k)\ \gamma^{(s)}(k) \quad , \quad k=1, \ldots, r, \tag{E6.33}$$

$$Y(k) = \frac{1}{\left(p^{(s)}\right)' z(1)} \left(p^{(s)}\right)' z(k) \quad , \quad k=1, \ldots, r, \tag{E6.34}$$

and

$$x = x^{(s)} \tag{E6.35}$$

is a solution to the model (E6.1) – (E6.7).

Clearly our suggested solution satisfies (E6.2) – (E6.7). Does it satisfy (E6.1)? Under (E6.9), the solution of the utility maximizing problem for household k is the solution for c(k) in the problem of choosing non-negative values for $\gamma(k)$ and c(k) to

$$\text{maximize} \qquad 1_h'\ \gamma(k) \tag{E6.36}$$

subject to

$$a(k)\ \gamma(k) \le c(k) \tag{E6.37}$$

and

$$p'\ c(k) \le Y(k) \quad . \tag{E6.38}$$

Non-negative vectors $\gamma(k)$ and c(k) are a solution to this problem if and only if there exist $q(k) \geq 0$ and $\lambda(k) \geq 0$ such that

$$1_h' - \left(q(k)\right)' a(k) \le 0 \quad , \tag{E6.39}$$

$$\left(1_h' - \left(q(k)\right)' a(k)\right) \gamma(k) = 0 \quad , \tag{E6.40}$$

$$\left(q(k)\right)' - \lambda(k) p' \le 0 \quad , \tag{E6.41}$$

$$\left(\left(q(k)\right)' - \lambda(k) p'\right) c(k) = 0 \quad , \tag{E6.42}$$

$$a(k)\ \gamma(k) - c(k) \le 0 \quad , \tag{E6.43}$$

$$\left(q(k)\right)' \left(a(k)\ \gamma(k) - c(k)\right) = 0 \quad , \tag{E6.44}$$

$$p'\ c(k) - Y(k) \le 0 \tag{E6.45}$$

and

$$\lambda(k) \left(p'\ c(k) - Y(k)\right) = 0 \quad . \tag{E6.46}$$

If p and Y(k) are given by (E6.32) and (E6.34), and (E6.31) is valid, then $\gamma(k) = \gamma^{(s)}(k)$ and $c(k) = a(k)\ \gamma^{(s)}(k)$ is a solution to (E6.36) – (E6.38); conditions (E6.39) – (E6.46) are satisfied with

$$q(k) = \delta^{(s)}(k)\ p^{(s)}$$

and

$$\lambda(k) = \left(p^{(s)}\right)'\ z(1)\ \delta^{(s)}(k) \quad .$$

We may conclude that the suggested solution (E6.32) – (E6.35) is consistent with the utility maximizing condition (E6.1) provided that (E6.31) holds.

The only remaining problem is how to set the $H^{(s)}(k)$, k=2, ..., r, in (E6.22) so that condition (E6.31) is satisfied. We might start with

$$H^{(1)}(k) = 0 \quad , \quad k=2, \ldots, r \quad .$$

This will ensure that problem (E6.21) – (E6.23) is feasible. Then we can adopt the rule

$$H^{(s)}(k) = \frac{\left(p^{(s-1)}\right)' z(k)}{\left(p^{(s-1)}\right)' a(k)\ \ \gamma^{(s-1)}(k)}\ H^{(s-1)}(k)\ , \quad k=2\ \ldots, r, \qquad \text{(E6.47)}$$

$H^{(s)}(k)$ is the utility level which household k can afford at the commodity prices, $p^{(s-1)}$, revealed from the solution to the $(s-1)^{th}$ linear program. When we obtain

$$H^{(s)}(k) = H^{(s-1)}(k) \quad , \quad k=2, \ldots, r \quad ,$$

condition (E6.31) is satisfied and we have found a solution to the model (E6.1) – (E6.7). In practice, convergence is usually very fast, although not guaranteed. It is possible that the $H^{(s)}(k)$s could be set so that problem (E6.21) – (E6.23) is infeasible. If this difficulty occurred in a practical example, rule (E6.47) could be modified to produce more gradual adjustments in the H(k)s. For example, we might use

$$H^{(s)}(k) = H^{(s-1)}(k) + \frac{1}{2}\left(\frac{\left(p^{(s-1)}\right)' z(k)}{\left(p^{(s-1)}\right)' a(k)\ \ \gamma^{(s-1)}(k)} - 1\right) H^{(s-1)}(k)\ , \quad k=2, \ldots, r \quad .$$

Under this rule, the adjustments in the H(k)s are half those implied by (E6.47).

### *Exercise 7 An introduction to the specialization problem in models of small open economies*

Consider an economy in which an equilibrium is a list of vectors and scalars $\{p, \gamma, x, \theta, g\}$, all apart from g being non-negative, satisfying

$$\gamma a \le z + Ax + \begin{pmatrix} g \\ 0 \end{pmatrix} \qquad \text{(E7.1)}$$

$$p'\left(\gamma a - z - Ax - \begin{pmatrix} g \\ 0 \end{pmatrix}\right) = 0 \qquad \text{(E7.2)}$$

$$p'A \le 0 \qquad \text{(E7.3)}$$

$$p'Ax = 0 \qquad \text{(E7.4)}$$

$$p_i = \theta p_i^w , \qquad i=1, \ldots, v, \qquad \text{(E7.5)}$$

$$\left(p^w\right)' g \le 0 , \qquad \text{(E7.6)}$$

$$\theta (p^w)' g = 0 \qquad \text{(E7.7)}$$

and

$$p'a = 1 . \qquad \text{(E7.8)}$$

The endogenous variables are p, the $n \times 1$ vector of commodity prices on the home market; $\gamma$, the scalar indicating the number of commodity bundles consumed; x, the $m \times 1$ vector of production activity levels; $\theta$, the exchange rate ($domestic/$foreign) and g, the $v \times 1$ vector of net imports (imports minus exports). If the $i^{th}$ component of g is positive, then the economy is a net importer of good i. If the $i^{th}$ component is negative, then the economy is a net exporter of i. Because not all commodities enter international trade, v is less than n. The first v commodities are traded goods and the last $n - v$ are non-traded. Primary factors (e.g., labour) are usually included among the non-traded commodities.

The exogenous variables are a, the $n \times 1$ vector giving the commodity composition of the consumption bundle; z, the $n \times 1$ vector giving the economy's resource endowments; A the $n \times m$ production technology matrix and $p^W$, the $v \times 1$ vector of prices on world markets of traded commodities. By making these prices exogenous, we have assumed that the economy under consideration is a "small country". That is, we have assumed that the economy is not sufficiently large that changes in its volumes of exports and imports will affect world commodity prices.

With the inclusion of international trade in our model, the market clearing equations, (E7.1) – (E7.2), allow three sources of supply: the resource endowment, production and net imports.[20] The zero-pure-profits conditions, (E7.3) – (E7.4), are as in earlier exercises. Condition (E7.5) says that domestic prices for traded commodities are determined by translating world prices into domestic currency via the exchange rate. If the exchange rate is \$0.9 domestic per \$1 foreign and $p_i^w$ = \$10 foreign, then the local price, $p_i$, for commodity i will be \$9 domestic. No one in the local economy would be willing to pay more than \$9 domestic for a unit of good i since it can be obtained for that price through importing. On the other hand, no one in the local economy would be willing to sell a unit of good i for less than \$9 domestic since this much could be obtained through exporting. Of course, (E7.5) assumes that there are no transport costs, import duties or other impediments to trade. Conditions (E7.6) and (E7.7) are the market clearing equations for foreign currency. Net demand for foreign currency (the balance of trade deficit) must be less than or equal to zero. If net demand is negative, then the price of foreign currency (θ) will be zero. Condition (E7.8) sets the absolute price level.

(a) For $(\bar{p}, \bar{\gamma}, \bar{x}, \bar{\theta}, \bar{g})$ satisfying (E7.1) – (E7.8) show that

$$\bar{p}'\bar{\gamma}a = \bar{p}'z \quad .$$

That is, show that final expenditure equals income.

(b) Assume that

$$a = \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix} , \quad z = \begin{pmatrix} 0 \\ 0 \\ 200 \end{pmatrix} , \quad A = \begin{pmatrix} 1.0 & -0.2 \\ -0.4 & 1.0 \\ -1.0 & -2.0 \end{pmatrix}$$

and

$$p^w = \begin{pmatrix} 2 \\ 1 \end{pmatrix} .$$

Find the equilibrium values for p, γ, x, θ and g.

(c) Write a linear programming problem which would be a suitable vehicle for solving (E7.1) – (E7.8).

20 The "0" appearing in the net-import terms of (E7.1) and (E7.2) is a vector of n – v zeros.

(d) Show that if the model (E7.1) – (E7.8) has a solution, then it has a solution in which no more than n – v components of x are non-zero. In an empirical application of the model, would you expect to find solutions in which more than n – v of the production activities were operated at positive levels?

(e) Models similar to (E7.1) – (E7.8) often imply unrealistic levels of industrial specialization. Discuss this in the light of your finding in part (d). What real world factors explaining industrial diversification are left out of the model (E7.1) – (E7.8)? No answer is provided for this question. In checking your answer, you might find it useful to read Dixon and Butlin (1977, section III, pp. 342-344).

***Answer to Exercise 7***

**(a)** From (E7.2) and (E7.5) we have

$$\bar{p}'\bar{\gamma}a - \bar{p}'z - \bar{p}'A\bar{x} - \bar{\theta}(p^w)'\bar{g} = 0 .$$

On using (E7.4) and (E7.7), we obtain the required result.

**(b)** Equilibrium prices must satisfy

$$(p_1, p_2, p_3)\begin{pmatrix} 1.0 & -0.2 \\ -0.4 & 1.0 \\ -1.0 & -2.0 \end{pmatrix} \le (0, 0) . \qquad (E7.9)$$

This can be reduced to[21]

$$(2, 1, p_3/\theta)\begin{pmatrix} 1.0 & -0.2 \\ -0.4 & 1.0 \\ -1.0 & -2.0 \end{pmatrix} \le (0, 0) . \qquad (E7.10)$$

We assume that at least one of the two production activities is operated at a non-zero level. Thus we look for a value for $p_3/\theta$ such that

21 We may assume that $\theta \neq 0$. If $\theta$ were zero, then $p_1$ and $p_2$ would be zero. Given that $a' = (3, 2, 0)$, we would have $p'a = 0$, violating (E7.8).

$$1.6 - p_3/\theta \leq 0$$

and

$$0.6 - 2(p_3/\theta) \leq 0 \quad ,$$

with at least one of these conditions holding as an equality. The only value compatible with these requirements is

$$(p_3/\theta) = 1.6 \quad , \tag{E7.11}$$

implying that only activity 1 will be operated at a positive level. Because $p_3 > 0$, we can be sure that demand equals supply for commodity 3. Hence, activity 1 must be operated at the 200 level. Therefore

$$x' = (200, 0) \quad .$$

Commodity prices and the exchange rate can be determined from (E7.5), (E7.8) and (E7.11), giving

$$\theta = 0.125 \text{ and } p' = (0.250, 0.125, 0.200) \quad .$$

$\gamma$ can be computed as

$$\gamma = p'z/p'a = (0.2)(200)/1 = 40 \quad .$$

Finally, we determine g from

$$40\begin{pmatrix}3\\2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} + \begin{pmatrix}1.0 & -0.2\\-0.4 & 1.0\end{pmatrix}\begin{pmatrix}200\\0\end{pmatrix} + \begin{pmatrix}g_1\\g_2\end{pmatrix} \quad ,$$

obtaining

$$g' = (-80, 160) \quad .$$

**(c)** Consider the problem of choosing values for $\gamma$, x and g, with $\gamma$ and x being non-negative, to

$$\left.\begin{aligned} &\text{maximize} \quad \gamma \\ &\text{subject to} \\ &\gamma a \leq z + Ax + \begin{pmatrix}g\\0\end{pmatrix} \\ &\text{and} \\ &(p^w)'g \leq 0. \end{aligned}\right\} \tag{E7.12}$$

If $(\bar{\gamma}, \bar{x}, \bar{g})$ is a solution to this problem, then there exists $\bar{p} \geq 0$ and $\bar{\theta} \geq 0$ such that

$$1 - \bar{p}' a = 0 \quad , \qquad \text{(E7.13)[22]}$$

$$\bar{p}' A \leq 0 \quad , \qquad \text{(E7.14)}$$

$$\bar{p}' A\bar{x} = 0 \quad , \qquad \text{(E7.15)}$$

$$\bar{p}_i = \bar{\theta} p_i^w \quad , \quad i=1, \ldots, v, \qquad \text{(E7.16)[23]}$$

$$\bar{\gamma} a \leq z + A\bar{x} + \begin{pmatrix} \bar{g} \\ 0 \end{pmatrix} \quad , \qquad \text{(E7.17)}$$

$$\bar{p}' \left( \bar{\gamma} a - z - A\bar{x} - \begin{pmatrix} \bar{g} \\ 0 \end{pmatrix} \right) = 0 \quad , \qquad \text{(E7.18)}$$

$$(p^w)' \bar{g} \leq 0 \qquad \text{(E7.19)}$$

and

$$\bar{\theta} (p^w)' \bar{g} = 0 \quad . \qquad \text{(E7.20)}$$

Thus $\{\bar{p}, \bar{\gamma}, \bar{x}, \bar{\theta}, \bar{g}\}$ satisfies the conditions (E7.1) – (E7.8). This means that solutions for the model (E7.1) – (E7.8) can be computed by solving the linear programming problem (E7.12), together with the associated dual problem.

Alternatively, we could solve a smaller linear programming problem:

choose non-negative values for $\gamma$ and $x$

to maximize $\gamma$

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22 We assume that $\bar{\gamma} > 0$.

23 Remember that g is not restricted with respect to sign. Consequently (E7.16) is an equality.

$$\left.\begin{aligned} &\text{subject to} \\ &\gamma\, (p^w)' a_T \le (p^w)' z_T + (p^w)' A_T x \\ &\text{and} \\ &\gamma\, a_N \le z_N + A_N x, \end{aligned}\right\} \qquad \text{(E7.21)}$$

where $a_T$, $z_T$ and $A_T$ are the first v rows of a, z and A, i.e., the rows referring to the traded goods, and $a_N$, $z_N$ and $A_N$ are the last n – v rows, i.e., the rows referring to the non-traded goods. In forming problem (E7.21), we have eliminated the g vector from problem (E7.12). If $(\bar{\gamma}, \bar{x})$ is a solution to (E7.21), there exist $\bar{p}_N \ge 0$ and $\bar{\theta} \ge 0$, such that

$$1 - \bar{\theta}(p^w)' a_T - \bar{p}_N' a_N = 0 \qquad \text{(E7.22)[24]}$$

$$\bar{\theta}(p^w)' A_T + \bar{p}_N' A_N \le 0 , \qquad \text{(E7.23)}$$

$$\left(\bar{\theta}(p^w)' A_T + \bar{p}_N' A_N\right)\bar{x} = 0 , \qquad \text{(E7.24)}$$

$$\bar{\gamma}\ (p^w)' a_T \le (p^w)' z_T + (p^w)' A_T \bar{x} , \qquad \text{(E7.25)}$$

$$\bar{\theta}\left(\bar{\gamma}(p^w)' a_T - (p^w)' z_T - (p^w)' A_T \bar{x}\right) = 0 , \qquad \text{(E7.26)}$$

$$\bar{\gamma}\, a_N \le z_N + A_N \bar{x} \qquad \text{(E7.27)}$$

and

$$\bar{p}_N' \left(\bar{\gamma} a_N - z_N - A_N \bar{x}\right) = 0 . \qquad \text{(E7.28)}$$

Having solved problem (E7.21) and its dual, we can compute $\bar{g}$ and $\bar{p}_T$ according to

$$\left.\begin{aligned} &\bar{g} = \bar{\gamma}\, a_T - z_T - A_T \bar{x} \\ &\text{and} \\ &\bar{p}_T = \bar{\theta}\, p^w . \end{aligned}\right\} \qquad \text{(E7.29)}$$

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24 Again we assume that $\bar{\gamma} > 0$.

We see that $\{(\bar{p}_T', \bar{p}_N')', \bar{\gamma}, \bar{x}, \bar{\theta}, \bar{g}\}$ is a solution to the model (E7.1) – (E7.8). Hence the model can be solved by using the linear programming problem (E7.21) and the post-solution computations, (E7.29).

**(d)** Let $\{\bar{p}, \bar{\gamma}, \bar{x}, \bar{\theta}, \bar{g}\}$ be a solution to the model (E7.1) – (E7.8). Then the linear programming problem (E7.21) has a solution. In fact $(\bar{\gamma}, \bar{x})$ is a solution.[25] Since (E7.21) has a solution, it must have a basic solution.[26] The number of non-zero variables in a basic solution is no more than the number of constraints. Thus, problem (E7.21) has a solution with no more than 1 + n – v variables at non-zero levels. We can assume that solutions to (E7.21) have a positive value for $\gamma$. We conclude that (E7.21) has a solution in which no more than n – v of the components of x are non-zero. Because any solution to (E7.21), together with the associated dual solution and the post-solution computations (E7.29), provides a solution to the model (E7.1) – (E7.8), we have shown that if the model has a solution then it has a solution in which no more than n - v of the production activities are operated.

In an empirical application of the model (E7.1) – (E7.8) it would be considered an unlikely accident if there were any solutions with more than n– v production activities at non-zero levels. To see why, consider the problem of finding values for $\bar{\theta}$ and $\bar{p}_N$ to satisfy

$$(\bar{\theta}, \bar{p}_N') \begin{bmatrix} (p^w)'A_T^* & (\bar{p}^w)'a_T \\ A_N^* & a_N \end{bmatrix} = (0', 1) \qquad \text{(E7.30)}$$

where A* is formed by selecting columns of A and $A_T^*$ consists of the first v rows of A* and $A_N^*$ consist of the remaining (n – v) rows. If A* contains more than n – v columns, then the system (E7.30) has more equations than variables. In this circumstance, it would be surprising if it had a solution. It is reasonable to assume that (E7.30) has no solution for any choice of A* having more than n – v columns. Under this assumption, the model (E7.1) – (E7.8) could not have a solution with

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25 Conditions (E7.22) – (E7.28) are both necessary and sufficient for a solution of (E7.21). See parts 1 and 2 of the Appendix.

26 See parts 3 – 5 of the Appendix .

more than n – v non-zero production activities. Notice from (E7.3) – (E7.5) and (E7.8) that if $\bar{\theta}$ and $\bar{p}_N$ are part of a solution to the model (E7.1) – (E7.8), then they satisfy (E7.30) where the columns of A selected in forming A* are those of the non-zero production activities.

***Exercise 8 Tariffs, export subsidies and transport costs in a linear model of a small open economy***

Consider an economy in which an equilibrium is a list of non-negative vectors and scalars {p, γ, x, θ, m, e}, satisfying

$$\gamma a \le z + Ax + \binom{m}{0} - \binom{e}{0} \quad , \qquad \text{(E8.1)}$$

$$p'\left(\gamma a - z - Ax - \binom{m}{0} + \binom{e}{0}\right) = 0 \quad , \qquad \text{(E8.2)}$$

$$p' A \le 0 \quad , \qquad \text{(E8.3)}$$

$$p' Ax = 0 \quad , \qquad \text{(E8.4)}$$

$$p_i \le p_i^m \theta (1 + T_i) \quad , \qquad i=1, \ldots, v, \qquad \text{(E8.5)}$$

$$\left(p_i - p_i^m \theta (1 + T_i)\right) m_i = 0 \ , \qquad i=1, \ldots, v, \qquad \text{(E8.6)}$$

$$p_i \ge p_i^e \theta(1 + S_i) \quad , \qquad i=1, \ldots, v, \qquad \text{(E8.7)}$$

$$\left(p_i - p_i^e \theta (1 + S_i)\right) e_i = 0 \ , \qquad i=1, \ldots, v, \qquad \text{(E8.8)}$$

$$(p^m)'m - (p^e)' e \le 0 \ , \qquad \text{(E8.9)}$$

$$\theta\left((p^m)'m - (p^e)'e\right) = 0 \qquad \text{(E8.10)}$$

and

$$p'a = 1 \ . \qquad \text{(E8.11)}$$

Compared with the last model ((E7.1) – (E7.8)), this model includes three additional sets of variables. First, trade flows are represented by two vectors, m and e, instead of one vector, g. m is the $v \times 1$ vector of imports and e is the $v \times 1$ vector of exports. Second, we have included in the model ad valorem tariffs ($T_i$) on imports and ad valorem subsidies ($S_i$) on exports. Third, we have distinguished between the foreign currency cost per unit of import of good i ($p_i^m$) and the foreign currency receipt per unit of export of good i ($p_i^e$). $p_i^m$ will normally exceed $p_i^e$ because of transport costs. If $p_i^w$ is the price of good i on a central world market, then $p_i^e$ and $p_i^m$ might be estimated as

$$p_i^e = p_i^w - f_i \qquad \text{(E8.12)}$$

and

$$p_i^m = p_i^w + f_i \qquad \text{(E8.13)}$$

where $f_i$ is the cost of freighting units of commodity i between the domestic economy and the central world market. The remaining notation in (E8.1) – (E8.11) is as in Exercise 7.

Of the equations, only (E8.5) – (E8.8) may be unfamiliar. Equation (E8.5) says that the cost of importing good i is a ceiling on the domestic price. If the foreign currency price ($p_i^m$) is \$10 foreign, the exchange rate, θ, is \$2 domestic/\$ foreign and the ad valorem tariff rate ($T_i$) is 50 per cent, then no one in the domestic economy will pay more than \$30 domestic (= 10 × 2 × 1.5) for a unit of good i. At this price, units of good i are available through importing. If the domestic price is below \$30, then, according to (E8.6), no units will be imported. Equation (E8.7) says that the return to exporters per unit of export of good i is a floor on the domestic price. If the foreign currency receipts ($p_i^e$) are \$8 foreign, the exchange rate is \$2 domestic/\$ foreign and the ad valorem rate of export subsidy ($S_i$) is 25 per cent, then no one in the domestic economy will sell a unit of good i for less than \$20 (= 8 × 2 × 1.25). This much can be obtained by exporting. If the domestic price is above \$20, then, according to (E8.7), no units of good i will be exported.

(a) For (E8.1) – (E8.11) to have an economically sensible solution, $p_i^e(1 + S_i)$ must not exceed $p_i^m(1 + T_i)$ for any i=1, ..., v. Why? In answering the remaining sections of this exercise, assume that $p_i^e(1 + S_i) < p_i^m(1 + T_i)$ for all i.

(b) For $\{\bar{p}, \bar{\gamma}, \bar{x}, \bar{\theta}, \bar{m}, \bar{e}\}$ satisfying (E8.1) – (E8.11) show that

$$\bar{p}'\bar{\gamma}a = \bar{p}'z + \bar{\theta}\sum_{i=1}^{v} T_i\, p_i^m \bar{m}_i - \bar{\theta}\sum_{i=1}^{v} S_i\, p_i^e \bar{e}_i \; . \qquad \text{(E8.14)}$$

That is, show that final expenditure equals the value of the resource endowment plus tariff revenue minus subsidy payments.

(c) Show that the model (E8.1) – (E8.11) implies that no commodity will simultaneously be exported and imported provided only that the exchange rate is non-zero.

(d) Assume that

$$a = \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}, \quad z = \begin{pmatrix} 0 \\ 0 \\ 200 \end{pmatrix}, \quad A = \begin{pmatrix} 1.0 & -0.2 \\ -0.4 & 1.0 \\ -1.0 & -2.0 \end{pmatrix}$$

$$p^m = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad p^e = \begin{pmatrix} 0.7 \\ 0.7 \end{pmatrix}, \quad \begin{pmatrix} T_1 \\ T_2 \end{pmatrix} = \begin{pmatrix} 0.0 \\ 0.25 \end{pmatrix},$$

and

$$\begin{pmatrix} S_1 \\ S_2 \end{pmatrix} = \begin{pmatrix} 0.25 \\ 0.0 \end{pmatrix}.$$

Find the equilibrium values for p, $\gamma$, x, $\theta$, m and e.

(e) Assume that the tariff on importing good 2 is removed, i.e., $T_2 = 0$. Find the new equilibrium.

(f) Now assume that the export subsidy is also removed, i.e., $S_2 = 0$. Find the free-trade equilibrium.

(g) Compare your answers to parts (d), (e) and (f). Discuss the effects of removing the tariff and the export subsidy.

(h) Write a linear programming problem which would be a suitable vehicle for solving (E8.1) – (E8.11) in the special case where all subsidies and tariffs are zero.

(i) For the general case (including tariffs and export subsidies) suggest how (E8.1) – (E8.11) might be solved by sequence of linear programming problems.

(j) An enthusiastic model builder spent six months assembling the data base for model (E8.1) – (E8.11) with the idea of investigating the resource-allocation effects of changes in his country's policy on tariffs and export subsidies. His data distinguished 103 goods (i.e., n = 103). He decided to treat the first 60 as traded (v = 60). The last three, he decided to treat as non-producible primary factors, labour, capital and land. For each of the 100 producible goods, he identified three techniques of production. Thus, the dimensions of his A matrix were an impressive 103 × 300.

To solve the model he applied the algorithm described in our answer to part (i) using an efficient linear programming package. The computations went without a hitch. By spending only a few dollars of his research grant at the computer centre, he generated model solutions for an enormous variety of tariff and subsidy policies.

Unfortunately his results seemed very unrealistic. He noted that his model never showed more than three commodities with positive export levels. In those solutions where there were three exports, there were no commodities which were both imported and domestically produced and none which were obtained by more than one production technique. In many solutions there were less than three exported commodities. In some of these, one or two commodities were both imported and domestically produced and occasionally there were one or two commodities produced by more than one technique. In a number of the solutions where there were less than three exports, one or two of the primary factors were left in excess supply implying that their wage rates (prices) were zero. Can you explain these results?

***Answer to Exercise 8***

**(a)** If $p_i^e(1 + S_i)$ were greater than $p_i^m(1 + T_i)$, then in any solution to the model θ would have to be zero. Otherwise (E8.5) and (E8.7) could not be satisfied. Thus, if our model had any solution at all, it would be a solution in which the domestic prices of all traded goods (goods 1, ..., v) were zero. Intuitively, the problem is that if $p_i^e(1 + S_i)$ exceeds $p_i^m(1 + T_i)$ and θ is greater than zero, then profits are available simply by importing units of good i and re-exporting them.

**(b)** From (E8.2), we have

$$\bar{p}'\gamma\bar{a} = \bar{p}'z + \bar{p}'A\bar{x} + \sum_{i=1}^{v} \bar{p}_i \bar{m}_i - \sum_{i=1}^{v} \bar{p}_i \bar{e}_i \quad .$$

On using (E8.4), (E8.6), (E8.8) and (E8.10), we obtain (E8.14).

**(c)** If we had a solution to the model (E8.1) – (E8.11) in which commodity i was simultaneously imported and exported, then from (E8.6) and (E8.8) we would have

$$p_i^m \, \theta(1 + T_i) = p_i^e \, \theta(1 + S_i) \quad . \qquad \text{(E8.15)}$$

We assume that our data is such that

$$p_i^m(1 + T_i) > p_i^e(1 + S_i) \quad ,$$

see part (a). Thus, our model can not imply that good i is both imported and exported unless it also implies that the exchange rate, θ, is zero.

**(d)** In looking for an equilibrium solution, we will investigate three possibilities:

(1) good 2 is exported implying that activity 2 is operated at a positive level and that good 1 is imported,[27]

(2) good 1 is exported implying that activity 1 is operated at a positive level and that good 2 is imported,

(3) neither good 1 nor good 2 is exported implying that neither is imported and that both activities 1 and 2 are operated at positive levels.

*Possibility 1*

In this case, domestic prices must satisfy

$$(p_1, p_2, p_3)\begin{pmatrix} -0.2 & 3 \\ 1.0 & 2 \\ -2.0 & 0 \end{pmatrix} = (0, 1) \quad , \qquad \text{(E8.16)}$$

$$p_1 = \theta \qquad \text{(E8.17)}$$

and

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[27] We can assume in this example that model solutions satisfy (E8.9) as an equality. Otherwise $p_1$ and $p_2$ would be zero, violating (E8.11).

$$p_2 = 0.7\theta \quad . \tag{E8.18}$$

Condition (E8.16) imposes zero profits in activity 2 and sets the absolute price level according to (E8.11). Conditions (E8.17) and (E8.18) recognize that commodity 1 is imported and commodity 2 is exported.

From (E8.16) – (E8.18) we find that

$$\theta = 0.227 , \quad p_1 = 0.227 , \quad p_2 = 0.159 \quad \text{and } p_3 = 0.057 \quad .$$

On these prices, however, activity 1 could be operated at positive profit:

$$\textit{Profit per unit of activity 1} = 0.227 - (0.159)\,(0.4) - 0.057$$

$$= 0.106 \quad .$$

We may conclude that there is no equilibrium solution with commodity 2 being exported.

*Possibility 2*

In this case, domestic prices must satisfy

$$(p_1, p_2, p_3)\begin{pmatrix} 1.0 & 3 \\ -0.4 & 2 \\ -1.0 & 0 \end{pmatrix} = (0, 1) \quad , \tag{E8.19}$$

$$p_1 = 0.7\theta(1.25)$$

and

$$p_2 = \theta(1.25) \quad .$$

This gives

$$\theta = 0.195, \quad p_1 = 0.171, \quad p_2 = 0.244 \quad \text{and} \quad p_3 = 0.073 \quad .$$

These prices are inconsistent with the condition that profits are non-positive per unit of activity 2:

$$\textit{profit per unit of activity 2} = -(0.171)\,(0.2) + 0.244 - (0.073)(2)$$

$$= 0.064 \quad .$$

We may conclude that there is no equilibrium solution with commodity 1 being exported.

*Possibility 3*

Because both activities are operated, prices must satisfy

$$(p_1, p_2, p_3)\begin{pmatrix} 1.0 & -0.2 & 3 \\ -0.4 & 1.0 & 2 \\ -1.0 & -2.0 & 0 \end{pmatrix} = (0, 0, 1) \quad ,$$

i.e.,

$$(p_1, p_2, p_3) = (0.183, 0.225, 0.094) \quad . \qquad \text{(E8.20)}$$

With neither good being exported (and therefore neither being imported) and with non-zero prices, (E8.1) and (E8.2) imply that

$$\begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}\gamma = \begin{pmatrix} 0 \\ 0 \\ 200 \end{pmatrix} + \begin{pmatrix} 1.0 & -0.2 \\ -0.4 & 1.0 \\ -1.0 & -2.0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad ,$$

i.e.,

$$x_1 = 69.4, \quad x_2 = 65.3 \quad \text{and} \quad \gamma = 18.8 \quad . \qquad \text{(E8.21)}$$

Finally, if domestic prices are to be given by (E8.20), then the exchange rate $\theta$ must be consistent with

$$\left.\begin{array}{l} 0.7\,\theta(1.25) \le 0.183 \le \theta \\ \text{and} \\ 0.7\,\theta \le 0.225 \le \theta(1.25) \quad , \end{array}\right\} \qquad \text{(E8.22)}$$

where the conditions (E8.22) specify the ceilings and floors on domestic prices of traded commodities provided by the availability of import supplies and export markets. $\theta$ will satisfy (E8.22) if and only if

$$\theta \geq \max\left\{0.183, \quad \frac{0.225}{1.25}\right\} = 0.183$$

and

$$\theta \leq \min\left\{\frac{0.183}{(0.7)(1.25)}, \quad \frac{0.225}{0.7}\right\} = 0.209 \quad .$$

In summary, the conditions for an equilibrium are satisfied if

$$0.183 \leq \theta \leq 0.209 \quad , \qquad \text{(E8.23)}$$

$$m = e = 0 \quad ,$$

and

$p$, $\gamma$ and $x$ are given by (E8.20) and (E8.21).

Equilibrium values for p, $\gamma$, x, m and e are uniquely determined in this example. On the other hand, equilibrium values for the exchange rate occur throughout the range specified by (E8.23).

**(e)** We investigate the three possibilities set out in our answer to (d).

*Possibility 1*

Possibility 1 produces no equilibrium solutions. The arithmetic here is exactly the same as that under possibility 1 of part (d).

*Possibility 2*

In this case, domestic prices must satisfy (E8.19) and

$$p_1 = 0.7\theta(1.25)$$

and

$$p_2 = \theta \quad .$$

This gives

$$\theta = 0.216, \quad p_1 = 0.189, \quad p_2 = 0.216 \quad \text{and} \quad p_3 = 0.102. \qquad \text{(E8.24)}$$

We note that these prices are consistent with profits per unit of operation of activity 2 being non-positive:

$$\textit{profit per unit of activity 2} = -(0.189)\,(0.2) + 0.216 - (0.102)\,(2.0)$$

$$= -0.026 \quad .$$

With prices and the exchange rate being non-zero and the operation of activity 2 and the level of exports of good 2 being zero, (E8.1), (E8.2) and (E8.10) imply that

$$\begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix} \gamma = \begin{pmatrix} 0 \\ 0 \\ 200 \end{pmatrix} + \begin{pmatrix} 1.0 \\ -0.4 \\ -1.0 \end{pmatrix} x_1 + \begin{pmatrix} -e_1 \\ m_2 \\ 0 \end{pmatrix}$$

and that

$$m_2 - 0.7e_1 = 0 \quad .$$

This gives

$$x_1 = 200, \quad \gamma = 14.6, \quad e_1 = 156.1, \quad m_2 = 109.3 \quad . \qquad \text{(E8.25)}$$

(E8.24) and (E8.25) together with the conditions $e_2 = m_1 = x_2 = 0$, are an equilibrium solution.

*Possibility 3*

If both activities are operated, then prices must be given by (E8.20) and the exchange rate must satisfy

$$\left.\begin{array}{l} 0.7\,\theta(1.25) \le 0.183 \le \theta \\ 0.7\,\theta \le 0.225 \le \theta . \end{array}\right\} \qquad \text{(E8.26)}$$

$\theta$ will satisfy (E8.26) if and only if

$$\theta \geq \max\{0.183\;,\;0.225\} = 0.225$$

and

$$\theta \leq \min\left\{\frac{0.183}{(0.7)(1.25)}, \frac{0.225}{0.7}\right\} = 0.209 \quad .$$

Since there is no value for $\theta$ consistent with these last two conditions, there is no equilibrium solution with both activities operated at positive levels. Hence, there is no equilibrium solution in which neither good is exported.

We conclude that the only equilibrium solution is that located under possibility 2, i.e.,

$$p = \begin{pmatrix} 0.189 \\ 0.216 \\ 0.102 \end{pmatrix}, \;\gamma = 14.6, \;\; x = \begin{pmatrix} 200 \\ 0 \end{pmatrix}, \;\; \theta = 0.216,$$

$$m = \begin{pmatrix} 0 \\ 109.3 \end{pmatrix}, \;\; e = \begin{pmatrix} 156.1 \\ 0 \end{pmatrix}.$$

**(f)** Again we investigate the three possibilities.

*Possibility 1*

As in parts (d) and (e), possibility 1 produces no equilibrium solutions.

*Possibility 2*

In this case, domestic prices must satisfy (E8.19) and

$$p_1 = 0.7\theta$$

and

$$p_2 = \theta \;.$$

This gives

$$\theta = 0.244, \quad p_1 = 0.171, \quad p_2 = 0.244 \quad \text{and} \quad p_3 = 0.073 \quad .$$

At these prices, activity 2 can be operated at a profit. Thus, possibility 2 produces no solutions in this example.

*Possibility 3*

We follow the answer to part (d) under possibility 3 to establish that prices must satisfy (E8.20) and quantities must satisfy (E8.21).

The exchange rate, θ, must be consistent with

$$\left.\begin{array}{c} 0.7\theta \le 0.183 \le \theta \\ \text{and} \\ 0.7\theta \le 0.225 \le \theta \end{array}\right\} \qquad \text{(E8.27)}$$

θ will satisfy (E8.27) if and only if

$$\theta \geq \max\{0.183\ ,\ 0.225\} = 0.225$$

and

$$\theta \leq \min\left\{\frac{0.183}{0.7}\ ,\ \frac{0.225}{0.7}\right\} = 0.261 \quad .$$

Thus, the conditions for an equilibrium are satisfied if

$$0.225 \leq \theta \leq 0.261 \quad ,$$

$$m = e = 0$$

and

p, γ and x are given by (E8.20) and (E8.21).

**(g)** Table E8.1 lists the solutions obtained in parts (d), (e) and (f). These illustrate two general propositions. First, the deterioration in welfare (measured by γ) as we move from the first solution to the second illustrates Lipsey and Lancaster's (1957) famous second-best result. The removal of one distortion (the tariff) does not necessarily improve welfare in a situation where other distortions (the subsidy) remain.[28] However, the removal of the subsidy in a situation where it is the

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28 Hazari (1978) is an authoritative and readable text on the welfare effects of policy changes in an open economy with distortions.

Table E8.1

*Model Solutions Under Various Rates of Tariff and Export Subsidy*

| Variable | Initial situation, $T_2$=.25, $S_1$=.25 | Situation with elimination of tariff, $T_2$=0, $S_1$=.25 | Situation with removal of tariff and subsidy (free trade), $T_1$=0, $S_2$=0 |
|---|---|---|---|
| $p_1$ | 0.183 | 0.189 | 0.183 |
| $p_2$ | 0.225 | 0.216 | 0.225 |
| $p_3$ | 0.094 | 0.102 | 0.094 |
| $\gamma$ | 18.8 | 14.6 | 18.8 |
| $x_1$ | 69.4 | 200 | 69.4 |
| $x_2$ | 65.3 | 0 | 65.3 |
| $\theta$ | [0.183, 0.209] | 0.216 | [0.225, 0.261] |
| $m_1$ | 0 | 0 | 0 |
| $m_2$ | 0 | 109.3 | 0 |
| $e_1$ | 0 | 156.1 | 0 |
| $e_2$ | 0 | 0 | 0 |

only distortion can be expected to improve welfare. Hence, $\gamma$ rises as we go from solution 2 to solution 3. The second proposition is that an x per cent tariff on all importables combined with an x per cent subsidy on all exportables is equivalent to having free trade, see, for example, Corden (1971, p. 119). Thus, it is not surprising that equilibria 1 and 3 are identical apart from the exchange rate.

Turning now to the details of Table E8.1, we see that our model predicts only small changes in commodity prices as we change tariffs and subsidies. On the other hand there are large movements in quantities. With the removal of the 25 per cent tariff on commodity 2, the share of the workforce[29] used in activity 1 goes from 34.7 per cent

29 We assume that the non-producible good (good 3) is labour.

to 100 per cent. The ratio of the value of exports to GDP[30] goes from 0 to 2.02 (i.e., (156.1)(0.189)/14.6) and consumption falls in real terms by 22.3 per cent (18.8 to 14.6). Consumption has fallen despite the small rise in the wage rate ($p_3$). This is because disposable factor income has been reduced by the need to provide for the export subsidy (see equation (E8.14)).

The volatility of the quantity results reflects the lack of diversifying phenomena in our model, see Exercise 7(e). The model omits terms of trade effects and this particular example includes no activity-specific non-producible factors, e.g. agricultural land, skilled labour or mines. The example has only one non-producible factor and this is assumed to be perfectly mobile across activities.

**(h)** Consider the linear programming problem of choosing non-negative values for $\gamma$, x, m and e to

$$\left.\begin{aligned} &\text{maximize} && \gamma \\ &\text{subject to} && \gamma a \le z + Ax + \begin{pmatrix} m \\ 0 \end{pmatrix} - \begin{pmatrix} e \\ 0 \end{pmatrix} \\ &\text{and} && (p^m)'m - (p^e)'e \le 0. \end{aligned}\right\} \quad \text{(E8.28)}$$

The list of non-negative vectors and scalars, $\{\bar{\gamma}, \bar{x}, \bar{m}, \bar{e}\}$, is a solution to this problem if and only if there exists $\bar{p} \ge 0$, $\bar{\theta} \ge 0$ such that

$$1 - \bar{p}'a = 0 \quad , \qquad \text{(E8.29)[31]}$$

$$\bar{p}'A \le 0 \quad , \qquad \text{(E8.30)}$$

$$\bar{p}'A\bar{x} = 0 \quad , \qquad \text{(E8.31)}$$

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30 With the trade balance assumed to be zero, see (E8.10), and with prices normalized accordingly to (E8.11), GDP in this model is the number of consumption bundles, i.e. GDP is $\gamma$.

31 We assume that $\bar{\gamma} > 0$ so that (E8.29) can be written as an equality.

$$\bar{p}_i - \bar{\theta}\, p_i^m \le 0 \quad , \qquad i=1, \ldots, v, \qquad \text{(E8.32)}$$

$$(\bar{p}_i - \bar{\theta}\, p_i^m)\bar{m}_i = 0 \quad , \qquad i=1, \ldots, v, \qquad \text{(E8.33)}$$

$$-\bar{p}_i + \bar{\theta} p_i^e \le 0 \quad , \qquad i=1, \ldots, v, \qquad \text{(E8.34)}$$

$$(-\bar{p}_i + \bar{\theta} p_i^e)\bar{e}_i = 0 \quad , \qquad i=1, \ldots, v, \qquad \text{(E8.35)}$$

$$\bar{\gamma}\, a - z - A\bar{x} - \begin{pmatrix} \bar{m} \\ 0 \end{pmatrix} + \begin{pmatrix} \bar{e} \\ 0 \end{pmatrix} \le 0 \quad , \qquad \text{(E8.36)}$$

$$\bar{p}' \left( \bar{\gamma} a - z - A\bar{x} - \begin{pmatrix} \bar{m} \\ 0 \end{pmatrix} + \begin{pmatrix} \bar{e} \\ 0 \end{pmatrix} \right) = 0 \quad , \qquad \text{(E8.37)}$$

$$(p_m)' \bar{m} - (p^e)' \bar{e} \le 0 \qquad \text{(E8.38)}$$

and

$$\theta\left((p_m)' \bar{m} - (p^e)' \bar{e}\right) = 0 \quad . \qquad \text{(E8.39)}$$

Thus, $\{\bar{p}, \bar{\gamma}, \bar{x}, \bar{\theta}, \bar{m}, \bar{e}\}$ is a solution to the linear programming problem (E8.28) and its dual if and only if it satisfies the conditions (E8.1) – (E8.11) for the special case where the $S_i$ and $T_i$ are all zeros. This means that all solutions for the free-trade version of the model (E8.1) – (E8.11) can be computed by solving the linear programming problem (E8.28) together with the associated dual problem.

**(i)** Consider the linear programming problem of choosing non-negative values for $\gamma$, x, m and e to maximize

$$\gamma - \sum_{i=1}^{v} p_i^m\, \psi\, T_i\, m_i + \sum_{i=1}^{v} p_i^e\, \psi\, S_i\, e_i \qquad \text{(E8.40)}$$

subject to

$$\gamma a \le z + Ax + \begin{pmatrix} m \\ 0 \end{pmatrix} - \begin{pmatrix} e \\ 0 \end{pmatrix} \qquad \text{(E8.41)}$$

and

$$(p^m)'m - (p^e)'e \le 0 \quad . \qquad \text{(E8.42)}$$

$\psi$ is an iterative variable. It is varied as we move from one linear programming problem to the next, but is held constant as we solve each problem. As will become apparent, $\psi$ is our guess of an equilibrium value for the exchange rate.

The list of non-negative vectors and scalars, $\{\bar{\gamma}, \bar{x}, \bar{m}, \bar{e}\}$, is a solution to the problem (E8.40) – (E8.42) if and only if there exist $\bar{p} \geq 0$ and $\bar{\theta} \geq 0$ such that

$$1 - \bar{p}'a = 0 \quad , \qquad \text{(E8.43)}$$ [32]

$$\bar{p}'A \leq 0 \quad , \qquad \text{(E8.44)}$$

$$\bar{p}'A\bar{x} = 0 \quad , \qquad \text{(E8.45)}$$

$$-p_i^m \psi T_i + \bar{p}_i - \bar{\theta} p_i^m \leq 0, \qquad i=1, \ldots, v, \qquad \text{(E8.46)}$$

$$\left(-p_i^m \psi T_i + \bar{p}_i - \bar{\theta} p_i^m\right) \bar{m}_i = 0, \qquad i=1, \ldots, v, \qquad \text{(E8.47)}$$

$$p_i^e \psi S_i - \bar{p}_i + \bar{\theta} p_i^e \leq 0 , \qquad i=1, \ldots, v, \qquad \text{(E8.48)}$$

$$\left(p_i^e \psi S_i - \bar{p}_i + \bar{\theta} p_i^e\right) \bar{e}_i = 0, \qquad i=1, \ldots, v, \qquad \text{(E8.49)}$$

$$\bar{\gamma} a - z - A\bar{x} - \begin{pmatrix} \bar{m} \\ 0 \end{pmatrix} + \begin{pmatrix} \bar{e} \\ 0 \end{pmatrix} \leq 0 \quad , \qquad \text{(E8.50)}$$

$$\bar{p}' \left(\bar{\gamma} a - z - A\bar{x} - \begin{pmatrix} \bar{m} \\ 0 \end{pmatrix} + \begin{pmatrix} \bar{e} \\ 0 \end{pmatrix}\right) = 0 \quad , \qquad \text{(E8.51)}$$

$$(p^m)'\bar{m} - (p^e)'\bar{e} \leq 0 \qquad \text{(E8.52)}$$

and

$$\bar{\theta}\left((p^m)'\bar{m} - (p^e)'\bar{e}\right) = 0 \quad . \qquad \text{(E8.53)}$$

By comparing (E8.43) – (E8.53) with (E8.1) – (E8.11) we see that $\{\bar{p}, \bar{\gamma}\ \bar{x}, \bar{\theta}, \bar{m}, \bar{e}\}$ is a solution to the model (E8.1) – (E8.11) if

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32 Again for convenience, we assume that $\bar{\gamma} > 0$. However, we should note that if $\psi$ were large, it would be possible that the optimal solution to (E8.40) – (E8.42) would imply zero consumption and large volumes of exports to take advantage of the subsidy term in the objective function (E8.40).

$$\psi = \bar{\theta} \ . \tag{E8.54}$$

This suggests that we can solve the model (E8.1) – (E8.11) as follows: first we guess an equilibrium value, $\psi^{(1)}$, for the exchange rate. Then we solve the linear programming problem (E8.40) – (E8.42) and its dual. If the value, $\bar{\theta}^{(1)}$, for the exchange rate emerging from this computation is equal (or sufficiently close) to $\psi^{(1)}$ then we have found a solution to our model (E8.1) – (E8.11). If $\psi^{(1)} \neq \bar{\theta}^{(1)}$, then we reset the iterative variable. A rule that is often effective in practice is

$$\psi^{(s)} = \bar{\theta}^{(s-1)} \ , \tag{E8.55}$$

i.e., we reset the iterative variable for the $s^{th}$ linear programming problem according to the value for the exchange rate emerging from the $(s-1)^{th}$ linear program. Of course other adjustment rules could be used, e.g.,

$$\psi^{(s)} = \psi^{(s-1)} + k\left(\bar{\theta}^{(s-1)} - \psi^{(s-1)}\right) \ , \tag{E8.56}$$

where k is greater than zero but less than one. Under (E8.56) we move the iterative variable only part of the way towards the value for the exchange rate emerging from our latest linear programming solution. A partial adjustment strategy would be appropriate if cycling became a problem. Cycling would occur under (E8.55) if, for example, we set $\psi^{(1)} = 1$ and obtained $\bar{\theta}^{(1)} = 4$, say. Then when we set $\psi^{(2)} = 4$, we obtain $\bar{\theta}^{(2)} = 1$. This situation is illustrated in Figure E8.1.

While the possibility of cycling cannot be dismissed, we have not found it to be a problem in applied work with models similar to (E8.1) – (E8.11). The reason is that the dual variable ($\bar{\theta}$) associated with

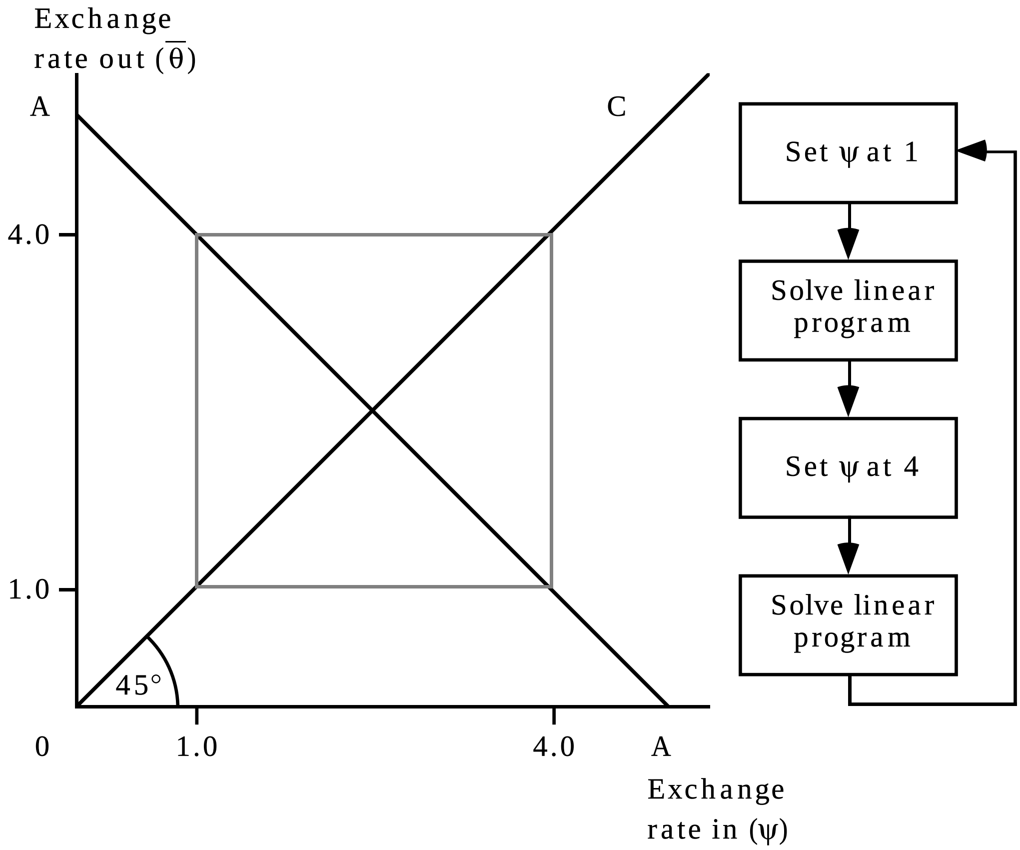


Figure E8.1 Cycling in an iterative procedure

AA shows the values for θ coming out of the linear programming problem (E8.40) - (E8.42) for different values of the iterative variable ψ. 0C is the 45 degree line. With the particular AA line shown here, if we set $\psi^{(1)} = 1$ and applied the adjustment procedure (E8.55), then we would obtain the never ending cycle shown in the right hand part of the figure. The cycle would be broken and convergence would occur quickly if we applied (E8.56) with k = 1/2.

constraint (E8.42) tends to be insensitive to the choice of the value for the iterative variable, $\psi$. In terms of Figure E8.1, the AA line is usually quite flat, ensuring rapid convergence of the iterative process (E8.55).

*Question:* Assume that

$$S_i = T_i = 0.20 \quad \text{for} \qquad i=1, \ldots, v. \qquad \text{(E8.57)}$$

Assume that problem (E8.40) – (E8.42) gives $\bar{\theta} = 1$ when we set $\psi = 0$. Draw the AA line. What is an equilibrium value for the exchange rate?

*Answer:* Under (E8.57), conditions (E8.46) – (E8.49) may be rewritten as

$$\bar{p}_i \le p_i^m \, (0.2\psi + \bar{\theta}) \quad , \qquad i=1, \ldots, v,$$

$$\left(\bar{p}_i - p_i^m (0.2\,\psi + \bar{\theta})\right) \bar{m}_i = 0 \quad , \qquad i=1, \ldots, v,$$

$$\bar{p}_i \ge p_i^e \, (0.2\psi + \bar{\theta}) \qquad i=1, \ldots, v,$$

and

$$\left(\bar{p}_i - p_i^e (0.2\,\psi + \bar{\theta})\right) \bar{e}_i = 0 \quad . \qquad i=1, \ldots, v.$$

With conditions (E8.46) – (E8.49) written in this form, it is apparent that if $\{\bar{p}, \bar{\gamma}, \bar{x}, \bar{\theta}(0), \bar{m}, \bar{e}\}$ satisfies (E8.43) – (E8.53) when $\psi = 0$, then $\{\bar{p}, \bar{\gamma}, \bar{x}, \bar{m}, \bar{e}\}$, together with

$$\bar{\theta} = \bar{\theta}(0) - 0.2\psi \quad ,$$

will satisfy (E8.43) – (E8.53) as we increase $\psi$. Thus, if $\bar{\theta}(0) = 1$, the AA line is that shown in Figure E8.2. An equilibrium value for the exchange rate is $\bar{\theta} = 0.83$.

*Question:* Would you expect the slope of AA always to be non-positive as in Figure E8.2?

*Answer:* $\bar{\theta}$ can be thought of as 1/h times the increase in the objective function (E8.40) which would be possible if we loosened the constraint (E8.42) to read

$$(p^m)'m - (p^e)'e \le h \quad , \qquad \text{(E8.58)}$$

where h is chosen to be a small positive number.[33] The availability of an extra h units of foreign exchange could be expected to allow increases in $\gamma$ via reductions in exports or increases in imports. In terms of the objective function (E8.40), both increases in imports and reductions in exports become less attractive as we increase $\psi$. Thus, we would normally expect a negative relationship between $\bar{\theta}$ and $\psi$.

**(j)** The model builder solved (E8.1) – (E8.11) using the linear programming problem (E8.40) – (E8.42) with an appropriate value for $\psi$. We assume that (E8.40) – (E8.42) has no non-basic solutions. (Even if there were some non-basic solutions, it is unlikely that the model builder would find them as most linear programming packages produce basic solutions only.) Thus, in his computed solutions the model builder will have no more than 104 (the number of constraints in (E8.41) – (E8.42)) of the 421 variables with non-zero values.[34]

We assume that $\gamma$ is one of the variables having a non-zero value. We also assume that model solutions involve non-zero supplies from production or importing of each of the 100 producible goods.[35] Hence, for each i=1, ..., 60, at least one of the four variables $x_{i1}$, $x_{i2}$, $x_{i3}$, $m_i$ must be non-zero where the $x_{ij}$ are the activity levels for the three i-producing activities and $m_i$ is the import level for good i. For i=61, ..., 100, at least one of the three variables $x_{i1}$, $x_{i2}$, $x_{i3}$ is non-zero. This

33 See, for example, Dixon, Bowles and Kendrick (1980, pp. 21-23).

34 The 421 variables are: $\gamma$, one variable; x, 300 variables; m, 60 variables and e, 60 variables.

35 This will certainly be true if $z_i = 0$, i=1, ..., 100. That is, the endowments of the producible goods are zero.

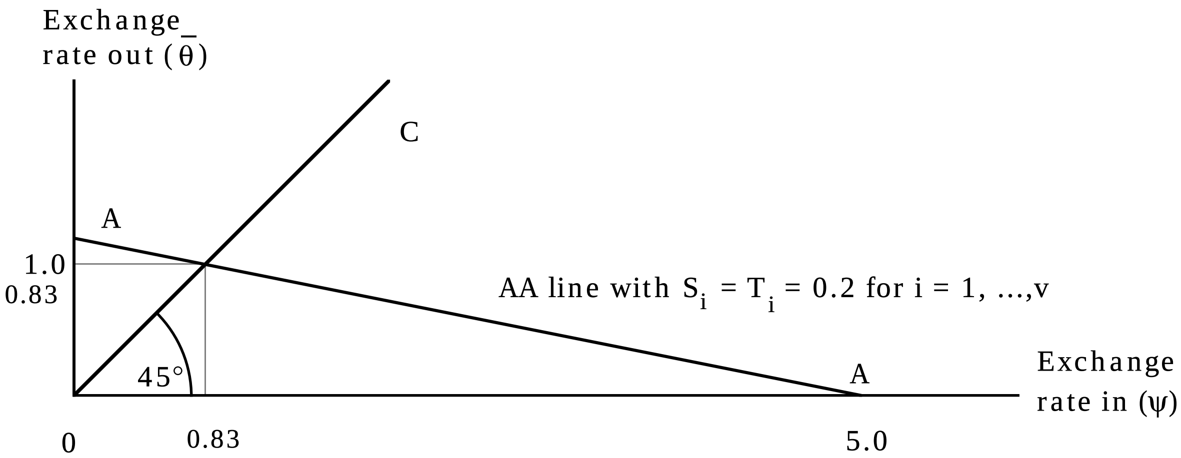


Figure E8.2 Exchange rate solution in a special case of the model (E8.1) – (E8.11)

Note that when $\psi$ reaches 5, $\bar{\theta}$ reaches zero. As $\psi$ increases further, $\bar{\theta}$ stays at zero and the optimal solutions for problem (E8.40) – (E8.42) involve no scarcity of foreign exchange. Exports are increased to take advantage of the export subsidy term in the objective function (E8.40) and imports are reduced to avoid the penalty from the tariff term.

leaves no more than three other production or trade variables with non-zero values. In particular, no more than three of the $e_i$ will be non-zero.

It is not necessary that there be three exported commodities. Two commodities might be exported and one commodity supplied from two sources, either two production activities or one production activity and imports. With two exported commodities, another possibility is that one primary factor could be left unemployed. In effect, the linear programming problem (E8.40) – (E8.42) would then have only 103 constraints implying that only 103 variables would appear in the solution with non-zero values. If only one commodity is exported, then two commodities could be supplied from two sources or one commodity could be supplied from three sources. Alternatively, one or two of the primary factors might be left in excess supply. A final possibility is that no commodities are exported. In this case, no commodities will be imported. Two primary factors (but not three) might be left in excess supply and one commodity supplied by two production techniques, etc.

### *Exercise 9 A long-run planning model with investment: the snapshot approach*

In the snapshot approach, we describe the economy at a particular point of time in the future. We do this without giving a detailed specification of the time paths of relevant variables. The idea is to construct a simple one period model which can be used to analyse the effects of demographic changes, technological progress and other long-run phenomena.

This exercise asks you to set up and analyse a typical (although simplified) snapshot model. You might use the exercise either before or after you do the reading from Manne (1963), Bruno (1967), Evans (1972) and Sandee (1960). If you choose the first route, you probably will not be able to do much of the exercise, but by studying the answer, you will get an introduction to the reading. The second route will enable you to check your grasp of the reading. Whichever route you take, you will notice in the reading that long-run planning models are often presented in the form of programming problems. In this chapter we have preferred to present models as sets of conditions defining equilibria. With this approach it is sometimes easier to identify the assumptions being made concerning the behaviour of the economic agents: the households, producers and investors.

(a) Imagine that as an economic planner you are asked in 1990 to forecast the output, investment, price and consumption structure of Linprogria for the year 2000. The economy of Linprogria is a simple one. There is no international trade and no government spending or taxes.

Using historical trends revealed from various statistical sources and information supplied by experts in each industry, you obtain a picture of the likely state of technology in 2000. You present this information in the form of an input-coefficients matrix, Q, a labour vector, *l*, and a capital matrix, K. The typical element of Q shows the input of good i which will be used up in the production of a unit of good j in 2000.[36] The typical element of *l* is the number of units of labour which will be needed per unit of output of good j and the typical element of K shows the quantity of good i which will be required in the creation of a unit of capital stock for industry j.[37] You define your capital units so that 1 unit of capital will be necessary to support a unit of production of good j and you assume that capital is not shiftable, i.e., units of capital created for industry j can only be used in industry j. Also, for simplicity you assume that capital lasts forever, i.e., that there is no depreciation.

Next, you consult with demographers and experts on labour participation rates to obtain a forecast, L, for the number of labour units which will be available in 2000. The demographers also explain to you the likely age and family structure of the 2000 population. You use this information in forming the vector, a, which describes the likely composition of household consumption in 2000. The typical element of a is the quantity of good i purchased by households per unit of household expenditure.

Finally, you make four assumptions concerning the behaviour of entrepreneurs: first that they are competitive so that there are no pure profits; second, that they do sufficient investment in each industry over the period 1990 to 2000 so that rates of return on industrial capital in 2000 are no more than r (10 percent, say) where r is the observed average rate of return on capital over some historical period, for example 1975

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36 In this exercise, producible goods and primary factors are treated separately. It is also assumed that there are n producible goods and that there is one technique or activity for producing each. Q is a non-negative $n \times n$ matrix referring to inputs of producible goods.

37 Industry j is the group of firms producing good j.

to 1990; third, that they do no investment in the year 2000 (or over the period 1990 to 2000) in industries having rates of return in 2000 of less than r; and fourth, that they increase the capital stock in each industry in 2000 sufficiently to maintain the average rate of growth of capital achieved in the industry from 1990 until 2000. This final assumption is a common trick in planning models. Investment in industry j in the final year, 2000, is tied down by relating it to the difference between the capital stock in industry j at the beginning of 2000 (as implied by the model) and the known current level of capital stock which we will denote by $k_j(90)$.

Specify your model of 2000 mathematically. You might set out your answer as follows:

The economy in 2000 is described by a list of non-negative scalars, vectors and matrices {x, p, γ, w, Π, J, k(00); Q, *l*, K, L, a, r, k(90)} satisfying the following set of equations and inequalities,

. . . . . (i)

. . . . . (ii)

.
.
.

where

- x is the vector of commodity (or industry) outputs;
- p is the vector of commodity prices;
- γ is a scalar showing the level of household consumption; i.e., the number of a-vectors consumed;
- w is the wage rate;
- Π is the vector of annual rental values of capital stock in each industry;
- J is the vector of increases in capital stocks (investment in 2000);
- k(00) is the vector of capital stocks by using industry at the beginning of 2000;
- (Q, *l*, K), L, a and r are data for 2000 describing technology, labour supply, consumer preferences and the economy-wide rate of return; and
- k(90) is the vector of capital stock by using industry at the beginning of 1990.

The equations and inequalities should specify that in 2000, demand will not exceed supply for commodities, labour and the capital stock of each industry; that the prices of commodities and primary factors in excess supply will be zero; that aggregate investment plus aggregate consumption expenditure will equal aggregate income; that commodity prices will not exceed costs (the zero pure profits assumption); that commodities whose prices fall short of costs will not be produced; that rates of return (i.e., annual rentals divided by costs of new capital) will not exceed the historically determined average rate r; that investment will be zero in industries having rates of return of less than r; and that investment will be sufficient to maintain the growth rates of capital in each industry implied by the model for 1990 to 2000.

(b) Check that the number of equation equivalents in your model is the same as the number of endogenous variables. What is the role of homogeneity and Walras' law?

(c) Suggest an algorithm for computing x, p, $\gamma$, w, $\Pi$, J, k(00) for a given data set Q, $l$, K, L, a, r, k(90). The algorithm should use a linear programming package at each step.

***Answer to Exercise 9***

**(a)** The list of endogenous variables

$$\Xi = \{x, p, \gamma, w, \Pi, J, k(00)\} \tag{E9.1}$$

will satisfy the equations and inequalities set out below in (E9.2) – (E9.8).

$$\left.\begin{aligned} &Qx + \gamma a + KJ \le x \\ \text{and} \quad & \\ &p'(Qx + \gamma a + KJ - x) = 0. \end{aligned}\right\} \tag{E9.2}$$

(E9.2) specifies that commodity demands (intermediate, household and investment) will not exceed supplies and can only be less than supplies for free goods.

$$\left.\begin{aligned} &l' x \le L \\ \text{and} \quad & \\ &w(l' x - L) = 0. \end{aligned}\right\} \tag{E9.3}$$

$$\left.\begin{aligned} & x \le k(00) \\ \text{and} \quad & \\ & \Pi'(x - k(00)) = 0. \end{aligned}\right\} \qquad \text{(E9.4)}$$

(E9.3) and (E9.4) specify that demands cannot exceed supplies of factors (labour and capital) and that if demand falls short of supply for any factor, then the relevant wage or rental will be zero.

$$p'\,KJ + \gamma p'\,a \;=\; \Pi'\,k(00) \;+\; wL \quad . \qquad \text{(E9.5)}$$

(E9.5) specifies that expenditure (investment plus consumption) equals income (rentals plus wages).

$$\left.\begin{aligned} & p' \le p'Q + w\,l' + \Pi' \\ \text{and} \quad & \\ & (p' - p'Q - w\,l' - \Pi')x = 0. \end{aligned}\right\} \qquad \text{(E9.6)}$$

(E9.6) specifies that prices cannot exceed costs and that if the price of good i is less than the cost, then the output of i will be zero.

$$\left.\begin{aligned} & \Pi_i / p'K_{.i} \le r \\ \text{and} \quad & \\ & J_i((\Pi_i / p'K_{.i}) - r) = 0 \quad , \text{for all } i. \end{aligned}\right\} \qquad \text{(E9.7)}$$

$K_{.i}$ is the $i^{th}$ column of K. Hence $p'K_{.i}$ is the cost of a unit of capital in industry i. $\Pi_i$ is its rental value. The rate of return is the ratio of rental to cost. If this ratio for industry i is less than the historically normal rate of return, r, then there will be no investment in industry i.

$$J_i / k_i(00) \;=\; (k_i(00)/k_i(90))^{0.1} \;-\; 1 \quad , \quad \text{for all } i \quad . \qquad \text{(E9.8)}$$

(E9.8) imposes the condition that investment in 2000 maintains for each industry the average rate of growth of capital stock for the period 1990 to 2000.[38]

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38 If you are having difficulty with (E9.8), consider the following equations:

$$k_i(01) \;=\; k_i(00) + J_i$$

$$k_i(00) \;=\; k_i(90)\,(1 + h_i)^{10}$$

and

$$k_i(01) \;=\; k_i(00)\,(1 + h_i) \;,$$

**(b)** Before moving to the question of computations, it is always reassuring to count equations and variables. Equation counting is not a complete substitute for a formal discussion of the existence question, but certainly if we detect a discrepancy between the number of equation equivalents in (E9.2) – (E9.8) and the number of variables in the list $\Xi$ (see (E9.1)) we will have doubts about either the completeness or internal consistency of our description of 2000.

Where n is the number of goods, we see that $\Xi$ contains n + n + 1 + 1 + n + n + n = 5n + 2 variables. Conditions (E9.2) – (E9.8) are equivalent to n + 1 + n + 1 + n + n + n = 5n + 2 equations.[39] But what about homogeneity and Walras' law? We consider homogeneity. It is clear that our description of 2000 says nothing about the absolute price level. If we have found a list $\Xi_1 = \{\bar{x}, \bar{p}, \bar{\gamma}, \bar{w}, \bar{\Pi}, \bar{J}, \bar{k}(00)\}$ which satisfies (E9.2) – (E9.8), then $\Xi_2 = \{\bar{x}, \lambda\bar{p}, \bar{\gamma}, \lambda\bar{w}, \lambda\bar{\Pi}, \bar{J}, \bar{k}(00)\}$ also satisfies (E9.2) – (E9.8) where $\lambda$ is any positive scalar. To remove the indeterminacy, we can fix the absolute price level by adding an equation such as

$$p'a = 1 \quad . \tag{E9.9}$$

(Another possibility would be w = 1). The addition of (E9.9) brings the number of equations to 5n + 3. However, Walras' law is applicable. The income-expenditure equation (E9.5) can be omitted. It can be derived from the other equations in the model as follows. From (E9.2) we have

$$\gamma p'a + p'KJ = p'x - p'Qx \quad .$$

On substituting from (E9.6) we obtain

$$\gamma p'a + p'KJ = wl'x + \Pi'x \quad .$$

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where $h_i$ is the average rate of growth of capital stock from 1990 to 2000 for industry i. Eliminate $k_i(01)$ and $h_i$ to obtain (E9.8).

39 (E9.2) is equivalent to n equations. For each of the n commodities, we either have demand equals supply or price equals zero. (E9.3) is equivalent to one equation, etc.

This reduces to (E9.5) via (E9.3) and (E9.4).

**(c)** The purpose of an algorithm is to break a difficult problem (the main problem) into a sequence of easy problems (the steps). Each of the step problems uses data revealed by the solutions to previous step problems. A good algorithm terminates rapidly with a final problem which reveals the answer to the main problem.

Designing an algorithm is a little like integration. Given the answer, it is not hard to see that it is in fact the answer. However, to find the answer in the first place often requires some ingenuity.

In this exercise, the main problem is to find values for x, p, $\gamma$, w, $\Pi$, J, and k(00) which satisfy conditions (E9.2) – (E9.8). This can be done by solving a sequence of linear programs where the $s^{th}$ program is to choose non-negative values for the scalar $\gamma$ and the vectors $x_1$ and $x_2$ to

$$\text{maximize} \quad \gamma \tag{E9.10}$$

subject to

$$-(I-Q)(x_1 + x_2) + \gamma a + rKx_2 - \chi^{(s)} \leq 0, \tag{E9.11}$$

$$l'(x_1 + x_2) - L \leq 0 \tag{E9.12}$$

and

$$x_1 - k(90) \leq 0 \quad . \tag{E9.13}$$

$\chi^{(s)}$ is a vector of iterative variables which is changed as we move from one linear program to the next. Its value is fixed exogenously in each program. Q, $l$, K, L, a, r, and k(90) are data and have already been defined.

We show that if $\chi^{(s)}$ has a suitable value, then the solution to the linear programming problem (E9.10) – (E9.13) reveals a solution to the system (E9.2) – (E9.8). To do this, we note that the list of non-negative vectors and scalars $\left(\gamma^{(s)}, x_1^{(s)}, x_2^{(s)}\right)$ is a solution to (E9.10) – (E9.13) if and only if there exist non-negative vectors and scalars $p^{(s)}$, $w^{(s)}$ and $\Pi^{(s)}$ such that

$$\left.\begin{aligned} -(I-Q)\left(x_1^{(s)}+x_2^{(s)}\right)+\gamma^{(s)}a+rKx_2^{(s)}-\chi^{(s)} &\leq 0, \\ (p^{(s)})'\left[-(I-Q)\left(x_1^{(s)}+x_2^{(s)}\right)+\gamma^{(s)}a+rKx_2^{(s)}-\chi^{(s)}\right] &= 0 \quad , \end{aligned}\right\} \tag{E9.14}$$

$$\left.\begin{array}{l} l'\left(x_1^{(s)}+x_2^{(s)}\right)-L\leq 0, \\ w^{(s)}\left[l'\left(x_1^{(s)}+x_2^{(s)}\right)-L\right]=0, \end{array}\right\} \quad \text{(E9.15)}$$

$$\left.\begin{array}{l} x_1^{(s)}-k(90)\leq 0, \\ \left(\Pi^{(s)}\right)'\left[\left(x_1^{(s)}-k(90)\right)\right]=0 \end{array}\right\} \quad \text{(E9.16)}$$

$$\left.\begin{array}{l} \left(p^{(s)}\right)'(I-Q)-w^{(s)}l'-\left(\Pi^{(s)}\right)'\leq 0, \\ \left[\left(p^{(s)}\right)'(I-Q)-w^{(s)}l'-\left(\Pi^{(s)}\right)'\right]x_1=0, \end{array}\right\} \quad \text{(E9.17)}$$

$$\left.\begin{array}{l} \left(p^{(s)}\right)'(I-Q)-w^{(s)}l'-r\left(p^{(s)}\right)'K\leq 0, \\ \left[\left(p^{(s)}\right)'(I-Q)-w^{(s)}l'-r\left(p^{(s)}\right)'K\right]x_2=0 \end{array}\right\} \quad \text{(E9.18)}$$

and

$$1-\left(p^{(s)}\right)'a = 0 \quad . \qquad \text{(E9.19)}[40]$$

If $\chi^{(s)}$ happens to have been chosen so that

$$\chi^{(s)} = rKx_2^{(s)} - KJ^{(s)} \qquad \text{(E9.20)}$$

where $J^{(s)}$ is a vector whose typical element is[41]

$$J_i^{(s)} = \left(x_{2i}^{(s)} + k_i(90)\right)\left[\left(\left(k_i(90)+x_{2i}^{(s)}\right)/k_i(90)\right)^{0.1} - 1\right] , \qquad \text{(E9.21)}$$

then it is not very hard to check that

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40 We assume that $\gamma^{(s)} > 0$.

41 We denote the $i^{th}$ element of J by $J_i$, the $i^{th}$ element of $x_2$ by $x_{2i}$, the $i^{th}$ element of k(90) by $k_i(90)$, etc.

$$\left.\begin{aligned} x &= x_1^{(s)} + x_2^{(s)} , \\ p &= p^{(s)} , \\ \gamma &= \gamma^{(s)} , \\ w &= w^{(s)} , \\ \Pi &= \Pi^{(s)} , \\ J &= J^{(s)} , \\ k(00) &= k(90) + x_2^{(s)} , \end{aligned}\right\} \qquad (E9.22)$$

is a solution to the system (E9.2) – (E9.8). For example, given that $x_1^{(s)}$, $x_2^{(s)}$ $\gamma^{(s)}$, $p^{(s)}$, $w^{(s)}$ and $\Pi^{(s)}$ satisfy (E9.14) and (E9.20) we can write

$$Q\left(x_1^{(s)}+x_2^{(s)}\right) + \gamma^{(s)}a + KJ^{(s)} \leq x_1^{(s)} + x_2^{(s)}$$

and

$$(p^{(s)})' \left[Q\left(x_1^{(s)}+x_2^{(s)}\right) + \gamma^{(s)}a+KJ^{(s)} - \left(x_1^{(s)}+x_2^{(s)}\right)\right] = 0 \quad ,$$

establishing that (E9.22) is consistent with (E9.2).

Perhaps the only difficulty is in establishing that (E9.22) satisfies (E9.7). Assume to the contrary that there exists i such that

$$\Pi_i^{(s)} \,/\, \left(p^{(s)}\right)' K_{.i} > r \quad , \qquad (E9.23)$$

i.e.,

$$\Pi_i^{(s)} > r(p^{(s)})' K_{.i} \quad .$$

Then, from (E9.18) we would have

$$(p^{(s)})' (I - Q)_{.i} - w^{(s)} l_i' - \Pi_i^{(s)} < 0 \quad ,$$

where $(I - Q)_{.i}$ is the $i^{th}$ column of $(I - Q)$. This would imply, via (E9.17), that

$$x_{1i}^{(s)} = 0 \quad .$$

This,via (E9.16), in turn would mean that

$$\Pi_i^{(s)} = 0 \tag{E9.24}$$

where we assume that $k_i(90)$ in (E9.16) is greater than zero. But (E9.24) is incompatible with (E9.23). Therefore

$$\Pi_i^{(s)} / (p^{(s)})' K_{.i} \leq r \text{ for all } i \quad . \tag{E9.25}$$

If

$$\Pi_i^{(s)} / (p^{(s)})' K_{.i} < r \quad , \tag{E9.26}$$

then by a similar argument to the one we have just made, we can show that

$$x_{2i}^{(s)} = 0 \quad .$$

This implies, via (E9.21), that

$$J_i^{(s)} = 0 \quad .$$

Thus, we have shown that

$$J_i^{(s)} \left[ \left( \Pi_i^{(s)} / (p^{(s)})' K_{.i} \right) - r \right] = 0 \text{ for all } i \quad . \tag{E9.27}$$

(E9.25) and (E9.27) together establish that (E9.22) satisfies (E9.7).

At this stage it might be helpful if we summarize the argument so far. Our problem is as follows: given data for Q, $l$, K, L, a, r and k(90), how can we compute an output vector x, a price vector p, a consumption level $\gamma$, a wage rate w, a rental vector $\Pi$, an investment vector J and a capital vector k(00) which satisfy our description, (E9.2) – (E9.8), of 2000? We have found that if we solve the programming problem, (E9.10) – (E9.13), (both primal and dual) and we happen to have chosen the appropriate value for the vector $\chi$, then we will have found what we were looking for. But how can we choose the right value for $\chi$? We can start by choosing an initial value $\chi^{(1)} = 0$, say. Then we can solve (E9.10) – (E9.13). This generates an $x_2^{(1)}$. Now we can revise our value of $\chi$ by setting $\chi^{(2)} = rKx_2^{(1)} - KJ^{(1)}$ where $J^{(1)}$ is computed according to (E9.21). We can re-solve (E9.10) – (E9.13) and generate a new value of $\chi$, $\chi^{(3)}$ etc. When $\chi^{(r)} \simeq \chi^{(r+1)}$ we can stop.

It is often rather difficult (and perhaps unnecessary) to prove convergence theorems for iterative processes such as the one we have just described. A little experimentation is probably more informative than theorizing. The important point to keep in mind in designing an algorithm is whether each step is revealing a good guess for the next step. It is often not sufficient merely to generate a direction of change for the iterative variables. A robust algorithm will also give an indication of step size.

## *Appendix* *BACKGROUND NOTES ON THE THEORY OF LINEAR PROGRAMMING*

### *A1 The standard linear programming problem*

The standard linear programming problem can be written as:

choose non-negative values for $x_1, \ldots, x_m$ to

$$\left.\begin{aligned} &\text{maximize} \quad \sum_{i=1}^{m} c_i x_i \\ &\text{subject to} \quad \sum_{i=1}^{m} a_i x_i \le b \end{aligned}\right\} \quad \text{(A1)}$$

where the $a_i$ and b are $n \times 1$ vectors.

### *A2 Necessary and sufficient conditions for the solution of the standard linear programming problem*

$\bar{x} = (\bar{x}_1, \ldots, \bar{x}_m)$ is a solution to (A1) if and only if there exists $\bar{p} \ge 0$ such that

$$\left.\begin{aligned} &c_i - \bar{p}' a_i \le 0, \quad i = 1, \ldots, m, \\ &(c_i - \bar{p}' a_i)\bar{x}_i = 0 \quad i = 1, \ldots, m, \\ &\sum_{i=1}^{m} a_i \bar{x}_i - b \le 0 \\ \text{and} \quad &\bar{p}' \left( \sum_{i=1}^{m} a_i \bar{x}_i - b \right) = 0. \end{aligned}\right\} \quad \text{(A2)}$$

If you need to review this proposition, look at Intriligator (1971, pp. 72-89) or Dixon, Bowles and Kendrick (1980, Exercise 1.14, pp. 47-51).

### *A3 Basic solutions*

Let $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_m)$ be a solution to (A1). Let

$$S(\bar{x}) = \{ i \mid \bar{x}_i > 0 \} \quad . \qquad \text{(A3)}$$

If there exist $\lambda_i$, $i \in S(\bar{x})$ such that not all the $\lambda_i$ are zero and such that

$$\sum_{i \in S(\bar{x})} a_i \lambda_i = 0 \quad ,$$

then $\bar{x}$ is not a basic solution. Otherwise it is a basic solution.

Notice that if $S(\bar{x})$ is empty, then $\bar{x}$ is a basic solution. Thus if $x = 0$ is a solution to (A1), it is a basic solution.

### *A4 Proposition on the existence of basic solutions*

If the linear programming problem (A1) has a solution, then it has a basic solution.

*Proof.*[42]: Let $\bar{x}$ be a solution to (A1), with $(\bar{x}, \bar{p})$ satisfying (A2). If $\bar{x}$ is a basic solution, then we have nothing to prove. If $\bar{x}$ is not a basic solution, then we can create another solution to (A1) which contains less non-zero activities than in our first solution. To do this, we note that if $\bar{x}$ is non-basic, then there exist $\lambda_i$, $i \in S(\bar{x})$, not all zero, such that

$$\sum_{i \in S(\bar{x})} a_i \lambda_i = 0 \quad . \qquad \text{(A4)}$$

We may assume that the $\lambda_i$ are chosen so that at least one of them is positive. Now we let

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42 This proof is derived from Gale (1960, Theorems 3.3 and 2.11).

$$\psi = \max_{i \in S(\bar{x})} \left\{ \frac{\lambda_i}{\bar{x}_i} \right\} .$$

$\psi$ is positive. Thus, the $\bar{\bar{x}}_i$s defined by

$$\bar{\bar{x}}_i = \frac{1}{\psi}\left(\psi - \frac{\lambda_i}{\bar{x}_i}\right) \bar{x}_i \quad \text{for} \quad i \in S(\bar{x}) \tag{A5}$$

are non-negative with at least one being zero. We let

$$\bar{\bar{x}}_i = 0 \quad \text{for} \quad i \notin S(\bar{x}) .$$

It is easy to check that $\bar{\bar{x}}_i$, $i=1, ..., m$ and $\bar{p}$ jointly satisfy (A2). This establishes that $\bar{\bar{x}} = (\bar{\bar{x}}_i , ..., \bar{\bar{x}}_m)$ is a solution to (A1). From the construction of $\bar{\bar{x}}$ we know that it contains fewer non-zero components than $\bar{x}$.

If $\bar{\bar{x}}$ is a basic solution, our search for a basic solution is finished. If not, we can construct a further solution to (A1) having fewer non-zero components than $\bar{\bar{x}}$. This process must terminate (possibly at $x = 0$). Thus we have shown that problem (A1) must have a basic solution if it has any solution.

*Corollary*. If problem (A1) has a solution, then it has a solution in which the number of non-zero activities is no more than n, the number of constraints.

*Proof:* Review Gale (1960, Theorem 2.1 and its corollaries, pp. 32-34).