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A General Welfare Decomposition for CGE models

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Abstract

Huff and Hertel (1996) derive a welfare decomposition for the equivalent variation in the GTAP model. The derivation appears to be very specific to GTAP. Nevertheless, it contains nearly all the ingredients required for performing welfare decomposition for any CGE model.

In this theoretical paper, the approach of Huff and Hertel (1996) is generalised to derive a welfare decomposition that can be applied to most, if not all, CGE models. General production and utility functions are accommodated, as are foreign income flows.

A brief guide to coding the proposed welfare decomposition in GEMPACK is also provided.

1 Introduction

In the GTAP model, economic welfare is represented as being derived from the allocation of national income between private consumption, government consumption and savings (Hertel 1997). This recognises that households gain benefits from their own current household consumption expenditure. They also benefit from current net national saving, since this increases their future household consumption.[1] Finally, they benefit from the government's provision of public goods and services, as proxied by current government expenditure.[2] National income is allocated between aggregate private consumption, aggregate government consumption and saving to maximise a top-level Cobb-Douglas utility function. With this functional form, successive increases in real household or government expenditure or saving generate equi-proportional increases in economic wellbeing. Aggregate private and government consumption are allocated between particular commodities to maximise constant difference elasticity (CDE) and Cobb-Douglas utility functions, respectively. As the CDE utility function is non-homothetic, this recognises that successive increases in private consumption of *particular* goods or services need not lead to equi-proportional increases in economic wellbeing.

Consequently, given such a definition of economic welfare, how well off a policy change actually makes a region depends on what the change does to its national income. It also depends on the effect of the policy change on prices, and hence the purchasing power of that income. Finally, it depends on how households evaluate the benefits of additional real expenditure. The last item — the marginal utility of real income — is a consequence of the assumed utility functions. National income is nominal net national product (NNP), and is equal to GDP less depreciation less net income payments to foreigners.

One particularly useful feature of GTAP that captures these dependencies is a *welfare decomposition* (Huff and Hertel 1996). This subdivides the overall measure of welfare into components that have a reasonably intuitive interpretation. As just noted, economic well-being depends in part on disposable income, which can be

[1] As noted in Hertel (1997), this derives from the work of Howe (1975), who showed that the intertemporal, extended linear expenditure system could be derived from an equivalent, atemporal maximisation problem, in which savings enters the utility function.

[2] As noted in Hertel (1997), this derives from the work of Keller (1980), who showed that if (1) preferences for public goods are separable from preferences for private goods, and (2) the utility function for public goods is identical across households, then a public utility function can be derived. The aggregation of this index with private utility to provide an overall welfare measure requires the further assumption that the level of public goods provided in the initial equilibrium is optimal.

divided into its components — GDP, depreciation, and net income payments to foreigners.[3] GDP can be further subdivided into the contributions from primary factors, net indirect taxes and technical changes. Decomposition along these lines leads to the following welfare contributions.

- *Endowment* contributions to welfare arise from changes in the availability of primary factors — for example, increases in the stock of machinery, buildings and agricultural land.
- *Technical efficiency* contributions arise from changes in the use of available inputs in production — for example, improvements in the labour productivity.
- *Allocative efficiency* contributions arise when the allocation of resources changes relative to pre-existing distortions.[4]

For any small change in the economy, allocative efficiency contributions are measured as the sum of a number of terms, where each term is the size of an initial indirect tax distortion, multiplied by the policy-induced change in the quantity of goods or services affected by that distortion.[5] The initial indirect tax distortion is the difference between the contribution to output from an additional unit of the good, and the price for which the good could be obtained in the absence of the tax. The product of the distortion and the change in the quantity therefore measures the net contribution to output from the change in the quantity of the good used. The allocative efficiency contribution for a large change to the economy equals the sum of the contributions for a sequence of small changes that are equivalent, in total, to the large change.

There are also contributions to national welfare arising from changes in relative prices (including export relative to import prices, or the terms of trade) as producers and consumers adjust their purchasing and sale patterns in response to policy change. There are also contributions to welfare arising from the likely flow on effects of production and terms of trade changes on foreign income flows. Finally, there is a (typically small) contribution arising because increases in real expenditure on particular goods and services may not lead to equi-proportional increases in economic wellbeing.

The derivation of the welfare decomposition in Huff and Hertel (1996) appears to be very specific to GTAP. It is even expressed in the TABLO notation of the

[3] Net income payments to foreigners are zero in GTAP.

[4] The GTAP welfare decomposition was motivated by the work of Keller (1980), which showed how the aggregate excess burden (the sum across households of compensating variations) was equal to allocative efficiency effects, in a model formulated for examining tax changes.

[5] In multi-step model simulations that correct for linearisation error, this can give an exact measure of the change in the welfare loss ‘triangle’ associated with a distortion.

GEMPACK software (Harrison and Pearson 1996) in which GTAP is implemented. Nevertheless, it contains nearly all the ingredients required for performing welfare decomposition for any CGE model. The derivation uses market clearing conditions for commodities and primary factors, and zero pure profit conditions for industries. These are relationships that would be present in most other CGE models.

In this paper the approach of Huff and Hertel (1996) is generalised to derive a welfare decomposition that can be applied to most, if not all, CGE models. The derivation only depends on assumptions that would normally be satisfied by most CGE models. There are six main differences between the approach adopted in this paper and that in Huff and Hertel (1996).

First, this paper decomposes the change in utility rather than a money metric measure of the change in welfare — such as the equivalent variation used in GTAP. Martin (1996) shows that money metric measures of changes in utility based on the expenditure and balance of trade functions differ for economies with existing taxes or subsidies. However, decomposition of a money metric measure of welfare would parallel the decomposition of utility described here, as is the case for the equivalent variation in GTAP.

Second, this paper includes welfare contributions from foreign income flows.

Third, the effect of non-homothetic preferences on welfare can be captured in a coefficient by which all the terms, the sum of which equals the change in utility, are multiplied. This is in contrast to GTAP where the effect of non-homothetic preferences on welfare is one of these terms — the variable described as the 'contribution to EV of marginal utility of income'.[6]

Fourth, the decomposition derived in this paper is general enough to cope with multi-product industries with non-separable inputs and outputs, and non-constant returns to scale in production.

Fifth, for each industry, terms measuring the welfare contributions caused by deviations from optimal or price taking behaviour, or from zero pure profits, are derived. They are proportional to the difference between indices of effective inputs and effective outputs for each industry.

Sixth, whereas in GTAP a nested utility function is assumed — NNP is allocated between aggregate private and government consumption and savings, and then the aggregate consumption is allocated across commodities — there is no requirement for a nested utility function in the current treatment.

[6] It should be emphasised that this is an *option*. If model users find it more helpful to retain such terms, that can easily be accommodated in the current framework.

Section 2 describes the conceptual economy for which the welfare decomposition is derived. The notation to be used and definitions are introduced, then the formal derivation is presented.

Two of the more difficult parts of the derivation are quarantined to section 3. First, the relationship between the NNP price index and a simple value-share-weighted index over component commodity prices is examined. This step is critical to the (optional) elimination, mentioned previously, of the GTAP marginal utility of income term in the welfare decomposition. Second, it is shown that the non-optimising/non-price taking welfare contribution is zero for industries that are revenue maximising, cost minimising and price taking, and have zero pure profits.

2 The formal derivation of the welfare decomposition

Consider an economy that consists of many activities, each of which uses various inputs. The inputs are divided into two groups — commodities and endowments. The activities are divided into two groups — industries, which produce (possibly multiple) commodities and use both commodities and endowments as inputs, and final demands, which do not produce anything and use only commodities as inputs. Each commodity may be produced by more than one industry. Taxes or subsidies may be levied on all inputs to all activities, and on all outputs from all industries.

The assumption that is fundamental in the derivation of the welfare decomposition is market clearing — the quantity of each commodity produced in the economy equals the total quantity of that commodity used in all activities.

Zero pure profits conditions — that the total cost of all inputs for each industry equals the total value of all commodities produced by that industry — are used later to eliminate some terms in the welfare decomposition (section 3), but are not essential to the derivation.

The market clearing condition applies to both domestic and imported commodities. The following convention is adopted to ensure that the condition applies for imports.[7] A final demand activity 'total imports of each commodity' is included, the inputs to which are imported commodities with negative values, equal in magnitude to the total CIF values of imported commodities used by all other activities. Therefore both the economy wide production and use of imported commodities is equal to zero; total imports count negatively in final demand; and a market clearing condition can be considered as applying to imported commodities.

[7] This convention is similar to how imports are shown in some input-output tables.

Other final demand activities include:

- total exports of each commodity;
- private consumption; and
- government consumption.

All other final demand activities will be called ‘investment’.[8]

Nominal national income, or NNP, is equal to the returns to all endowments (inclusive of income taxes), minus the value of depreciation of domestic capital, plus all indirect tax revenue, minus all indirect subsidy payments, plus net foreign income flows generated by a range of net foreign assets. Net foreign income flows may be positive or negative. Thus nominal NNP is equal to nominal GDP, minus depreciation, plus net foreign income flows.

Nominal NNP is allocated between purchases of private consumption commodities, government consumption commodities, and net (of depreciation) savings so as to maximise an utility function.

Motivation

Before introducing the notation and conventions required for the formal derivation, a brief overview of the derivation is now provided.

At the macro level, nominal NNP — which is equal to utility multiplied by the NNP price index — is split into GDP minus depreciation plus foreign income. The latter two items are then decomposed into nominal and real parts. The depreciation terms are written as a sum across industries, but could just as well have been left as a macro aggregate. The price indices of GDP and NNP, and any price parts of depreciation and foreign income, constitute the relative price contributions to welfare. Real GDP is then decomposed, in terms of the industry structure just outlined, into allocative efficiency, technical efficiency and endowment effects. It is at this stage that the market clearing conditions are critical. Finally, a residual term is obtained, which is zero if the conventional assumptions of CGE models — zero pure profits and optimising and price taking behaviour — are satisfied.

Consequently, quite a bit of notation is required to support the formal derivation — both its macro and micro components — and such notation is now introduced.

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[8] Thus all the final demands usually represented in an IO table and in the definition of GDP are present. Since, in some CGE models (for example, MONASH), there is an investment activity for each industry, it seemed sensible to allow for the possibility of many investment activities in the current treatment. It makes no difference in the formal derivation.

Notation

Upper case letters designate levels, lower case percentage changes. Δ means 'change in'. Superscripts on a symbol indicate to what item the symbol is related. Subscripts indicate a variety of types of the item indicated by the superscript. For example, Π_k^{DK} designates the asset price of domestic capital of type k.

Symbols used are:

- P — tax-inclusive price (rental price for assets), but tax-exclusive price when applied to industry outputs of commodities, that is, with superscript **0**;
- Π — asset price;
- Q — real quantity;
- V — tax-inclusive value (rental value for assets);
- $\hat{V}$ — tax-exclusive value (rental value for assets);
- R — tax revenue;
- T — advalorem tax rate;
- D — depreciation rate;
- $\Re$ — rate of return (lower case is $\tilde{n}$); and
- $U(\cdot)$ — indirect utility function governing the allocation of NNP.

A bar over a symbol indicates effective inputs (outputs). For quantities, these are the input quantities (output quantities) multiplied (divided) by the corresponding technical efficiencies. Effective prices are defined so that:

$$\overline{P}.\overline{Q} = P.Q$$

A bold, non-italicised symbol should be interpreted as a vector. For example, $\mathbf{P}^C$ is the vector of tax-inclusive prices of commodities purchased for private consumption. Multiplication of two vectors should be interpreted as the dot (that is, scalar) product. For example, the value of aggregate private consumption equals the sum of consumption prices times consumption quantities, thus: $V^C = \mathbf{P}^C\mathbf{.Q}^C$.

Superscripts used are:

- NNP — net national product;
- NDP — net domestic product;
- GDP — gross domestic product;
- DK — domestic capital;
- FY — foreign income;

- FA — foreign asset;
- DEP — depreciation;
- C — private consumption;
- G — government consumption;
- I — gross investment;
- X — exports;
- M — imports;
- S — savings;
- $\mathbf{I}$ — input into activities; and
- $\mathbf{O}$ — output from industries.

Subscripts used are:

- k — to range across types of domestic capital;
- $ö$ — to range across types of foreign assets;
- a — to range over activities;
- f — to range over final demands (a subset of activities);
- j — to range over industries (a subset of activities);
- i — to range over inputs;
- e — to range over endowments (a subset of inputs);
- c — to range over commodities (a subset of inputs); and
- $\bullet$ — total over a dimension.

Where two subscripts occur, the first refers to an element of the set of inputs, while the second refers to an element of the set of activities.

Definitions and conventions

Macro aggregates

Real NNP, Q^{NNP}, is defined to be the maximised value of utility, that is:

$$Q^{NNP} = U\left(\mathbf{P}^C, \mathbf{P}^G, P^S, V^{NNP}\right)$$

The price index of NNP, P^{NNP}, is defined as:

$$P^{NNP} = V^{NNP} / Q^{NNP}$$

The percentage changes in macro aggregates, except for NNP, are defined as value-share-weighted averages across all components. For example, aggregate real private consumption is defined by:

$$\begin{aligned} V^{C}.q^{C} &= \mathbf{V}^{C}.\mathbf{q}^{C} \\ &= \sum\nolimits_{c} V_{c}^{C}.q_{c}^{C} \end{aligned}$$

Again, if there is a set of investment activities $\Im$ (a subset of the set of final demands), then the investment price index is defined by:

$$\begin{aligned} V^{I}.p^{I} &= \mathbf{V}^{I}.\mathbf{p}^{I} \\ &= \sum\nolimits_{c,f\in\Im} V_{cf}^{I}.p_{cf}^{I} \end{aligned}$$

Foreign income

In reality, some foreign income flows — for example, foreign aid — are not returns to some asset. The convention adopted here is that such foreign income flows are returns on an asset, with the rate of return constant at one, and the asset price equal to the NNP price index, thus:

$$\text{Đ}_{\varphi}^{FA} = P^{NNP}$$
$$\Re_{\varphi}^{FA} = 1$$

The quantity of the asset is, consequently, the foreign income flow divided by the NNP price index. This is a sensible convention, with the foreign income flow being equal to the real foreign income flow times the 'price of utility'.

Tax-inclusive and tax exclusive prices and values

The next two equations relate the price received by industry j for producing commodity c (P_{cj}^{0}), the economy-wide uniform output tax inclusive price of commodity c (P_{c}^{0}), and the price paid by industry j for commodity c (P_{cj}^{I}).

$$P_{c}^{0} = P_{cj}^{0}.\left(1 + T_{cj}^{0}\right)$$
$$P_{cj}^{I} = P_{c}^{0}.\left(1 + T_{cj}^{I}\right)$$

The next four equations clarify the use of V and $\hat{V}$ to denote tax-inclusive and tax-exclusive values, respectively.

$$\hat{V}_{cj}^{0} = P_{cj}^{0}.Q_{cj}^{0}$$
$$V_{cj}^{0} = P_{c}^{0}.Q_{cj}^{0} = \hat{V}_{cj}^{0} + R_{cj}^{0}$$
$$\hat{V}_{cj}^{1} = P_{c}^{0}.Q_{cj}^{1}$$
$$V_{cj}^{1} = P_{cj}^{1}.Q_{cj}^{1} = \hat{V}_{cj}^{1} + R_{cj}^{1}$$

Derivation

Overview

The derivation proceeds, using linearised equations, as follows. Nominal NNP is split into GDP minus depreciation plus foreign income. The endowment and rate of return contributions to welfare from the latter two items are identified. The price index of GDP, the price index of NNP, and any asset price parts of depreciation and foreign income are manipulated to define terms of trade and asset price contributions to welfare. Then it only remains to decompose the percentage change in real GDP. Real GDP is expressed from the expenditure side as a share-weighted sum across commodity inputs into all final demand activities. Allocative efficiency contributions are derived by splitting off indirect tax revenues from the values of inputs and outputs multiplied by percentage changes in quantities. Market clearing conditions are used to eventually yield an expression that is a multiple of the difference between share-weighted indices of industries' outputs and inputs. These can be written as a weighted sum of technical efficiency terms — the technical efficiency contribution to welfare — and a difference of indices of effective outputs and effective inputs — the contribution due to non-optimising and/or non-price taking behaviour.

Formal derivation

Nominal NNP can be expressed as:

$$\begin{aligned} V^{NNP} &= V^{NDP} - V^{FY} \\ &= V^{GDP} - V^{DEP} + V^{FY} \\ &= V^{GDP} - \sum_k V_k^{DEP} + \sum_{\ddot{o}} V_{\ddot{o}}^{FA} \\ &= V^{GDP} - \sum_k D_k^{DK}.\text{Đ}_k^{DK}.Q_k^{DK} + \sum_{\ddot{o}} \Re_{\ddot{o}}^{FA}.\text{Đ}_{\ddot{o}}^{FA}.Q_{\ddot{o}}^{FA} \end{aligned}$$

The linearisation of this is:

$$V^{NNP}.\left(p^{NNP}+q^{NNP}\right)=V^{GDP}.\left(p^{GDP}+q^{GDP}\right)$$
$$-\sum_k V_k^{DEP}.\left(d_k^{DK}+\eth_k^{DK}+q_k^{DK}\right)$$
$$+\sum_{\ddot{o}} V_{\ddot{o}}^{FA}.\left(\tilde{n}_{\ddot{o}}^{FA}+\eth_{\ddot{o}}^{FA}+q_{\ddot{o}}^{FA}\right)$$

In section 3 it will be demonstrated that:

$$\left\{V^{NNP}.p^{NNP}-\left(V^{C}.p^{C}+V^{G}.p^{G}+V^{S}.p^{S}\right)\right\}+V^{NNP}.q^{NNP}$$
$$=\frac{V^{NNP}}{\partial\ln U\left(\mathbf{P}^{C},\mathbf{P}^{G},P^{S},V^{NNP}\right)/\partial\ln V^{NNP}}.q^{NNP}$$
$$\equiv \grave{E}.q^{NNP}$$

The term in braces is what gives rise to the 'marginal utility of income effect' in the GTAP welfare decomposition.[9] The above relationship allows this effect to be subsumed in the coefficient Θ in the current derivation. By the use of this relationship and the definition of the GDP price index:

$$V^{GDP}.p^{GDP}\equiv V^{C}.p^{C}+V^{G}.p^{G}+V^{I}.p^{I}+V^{X}.p^{X}-V^{M}.p^{M}$$

real NNP can be expressed as:

$$q^{NNP}=\frac{\left\{V^{NNP}.p^{NNP}-\left(V^{C}.p^{C}+V^{G}.p^{G}+V^{S}.p^{S}\right)\right\}+V^{NNP}.q^{NNP}}{\grave{E}}$$
$$\begin{aligned}
= \quad & V^{GDP}.q^{GDP}/\grave{E} && \\
& +\left[V^{X}.p^{X}-V^{M}.p^{M}\right]/\grave{E} && (=TOT)\\
& +\left[\left(V^{\acute{E}}.p^{\acute{E}}+\sum_{\ddot{o}}V_{\ddot{o}}^{FA}.\eth_{\ddot{o}}^{FA}\right)-\left(V^{S}.p^{S}+\sum_k V_k^{DEP}.\eth_k^{DK}\right)\right]/\grave{E} && (=ASS_PRI)\\
& +\sum_{\ddot{o}}V_{\ddot{o}}^{FA}.\tilde{n}_{\ddot{o}}^{FA}/\grave{E} && (=RORF)\\
& +\sum_{\ddot{o}}V_{\ddot{o}}^{FA}.q_{\ddot{o}}^{FA}/\grave{E} && (=FENDW)\\
& -\sum_k V_k^{DEP}.\left(d_k^{DK}+q_k^{DK}\right)/\grave{E} && (\rightarrow ENDW)
\end{aligned}$$

The terms in parentheses on the right-hand side show the correspondence between the components of real NNP and welfare contributions. An equal sign before the name of the welfare contribution indicates that the component of real NNP is equal to this contribution. An arrow indicates that the component of real NNP is one part of this contribution. The welfare contributions are:

- *TOT* — terms of trade;
- *ASS_PRI* — asset price;

[9] The term in the denominator of Θ is the elasticity of utility with respect to nominal income. It is one for a homothetic direct utility function. In that case, the price index of NNP is a share-weighted sum of component prices.

- *RORF* — foreign rate of return;
- *FENDW* — foreign endowment; and
- *ENDW* — (domestic) endowment contributions.

It now only remains to decompose the real GDP component isolated in the previous equation. Real GDP can be expressed, from the expenditure side, as:

$$
\begin{aligned}
V^{GDP}.q^{GDP} \big/ \grave{E} &= \sum\nolimits_{c,f} V_{cf}^{\mathrm{I}}.q_{cf}^{\mathrm{I}} \big/ \grave{E} \\
&= \sum\nolimits_{c,f} \hat{V}_{cf}^{\mathrm{I}}.q_{cf}^{\mathrm{I}} \big/ \grave{E} \\
&\quad + \sum\nolimits_{c,f} R_{cf}^{\mathrm{I}}.q_{cf}^{\mathrm{I}} \big/ \grave{E} \quad (\rightarrow ALLOC)
\end{aligned}
$$

The part of the allocative efficiency contribution (*ALLOC*) attributable to taxes on final demands has been identified. The remaining RHS term is:

$$
\begin{aligned}
\sum\nolimits_{c,f} \hat{V}_{cf}^{\mathrm{I}}.q_{cf}^{\mathrm{I}} \big/ \grave{E} &= \left[\sum\nolimits_{c,a} \hat{V}_{ca}^{\mathrm{I}}.q_{ca}^{\mathrm{I}} - \sum\nolimits_{c,j} \hat{V}_{cj}^{\mathrm{I}}.q_{cj}^{\mathrm{I}}\right] \big/ \grave{E} \\
&= \left[\sum\nolimits_{c,a} \hat{V}_{ca}^{\mathrm{I}}.q_{ca}^{\mathrm{I}} - \sum\nolimits_{i,j} \hat{V}_{ij}^{\mathrm{I}}.q_{ij}^{\mathrm{I}} + \sum\nolimits_{e,j} \hat{V}_{ej}^{\mathrm{I}}.q_{ej}^{\mathrm{I}}\right] \big/ \grave{E} \\
&= \left[\sum\nolimits_{c,a} \hat{V}_{ca}^{\mathrm{I}}.q_{ca}^{\mathrm{I}} - \sum\nolimits_{i,j} V_{ij}^{\mathrm{I}}.q_{ij}^{\mathrm{I}}\right] \big/ \grave{E} \\
&\quad + \sum\nolimits_{i,j} R_{ij}^{\mathrm{I}}.q_{ij}^{\mathrm{I}} \big/ \grave{E} \qquad (\rightarrow ALLOC) \\
&\quad + \sum\nolimits_{e,j} \hat{V}_{ej}^{\mathrm{I}}.q_{ej}^{\mathrm{I}} \big/ \grave{E} \qquad (\rightarrow ENDW)
\end{aligned}
$$

The first line is just accounting. The sum over final demand activities has been replaced by the difference between a sum over all activities and a sum over industries. The second line replaces this latter sum, which ranges over all commodity inputs into all industries, with a sum over all inputs into all industries. An offsetting sum — the third sum on the second line — of all endowment inputs into all industries is introduced. In the final RHS expression, indirect taxes have been added to the values in the sum over industry inputs. The offsetting sum over products of industry input-tax revenues and percentage changes in input quantities is that part of the allocative efficiency contribution (*ALLOC*) attributable to these taxes. The sum over endowment inputs into industries is part of the endowment contribution (*ENDW*). The first sum in the final RHS expression is dealt with using the market clearing conditions, which are now derived in linearised form. In levels, for all commodities *c*:

$$
\sum\nolimits_{a} Q_{ca}^{\mathrm{I}} = \sum\nolimits_{j} Q_{cj}^{\mathbf{0}}
$$

Linearisation yields:

$$
\sum\nolimits_{a} Q_{ca}^{\mathrm{I}}.q_{ca}^{\mathrm{I}} = \sum\nolimits_{j} Q_{cj}^{\mathbf{0}}.q_{cj}^{\mathbf{0}}
$$

and, multiplying by the price, P_c^0, of commodity c:

$$\sum_a \hat{V}^1_{ca}.q^1_{ca} = \sum_j V^0_{cj}.q^0_{cj}$$

Therefore,

$$\begin{aligned}
\left[\sum_{c,a} \hat{V}^1_{ca}.q^1_{ca} - \sum_{i,j} V^1_{ij}.q^1_{ij}\right] \Big/ \grave{E} &= \left[\sum_{c,j} V^0_{cj}.q^0_{cj} - \sum_{i,j} V^1_{ij}.q^1_{ij}\right] \Big/ \grave{E} \\
&= \left[\sum_{c,j} \hat{V}^0_{cj}.q^0_{cj} - \sum_{i,j} V^1_{ij}.q^1_{ij}\right] \Big/ \grave{E} \\
&\quad + \sum_{c,j} R^0_{cj}.q^0_{cj} \Big/ \grave{E} \\
&= \left[\sum_{c,j} \hat{V}^0_{cj}.\bar{q}^0_{cj} - \sum_{i,j} V^1_{ij}.\bar{q}^1_{ij}\right] \Big/ \grave{E} && (= NONOPT) \\
&\quad + \left[\sum_{c,j} \hat{V}^0_{cj}.a^0_{cj} + \sum_{i,j} V^1_{ij}.a^1_{ij}\right] \Big/ \grave{E} && (= TECH) \\
&\quad + \sum_{c,j} R^0_{cj}.q^0_{cj} \Big/ \grave{E} && (\rightarrow ALLOC)
\end{aligned}$$

In the second line, output tax revenues are taken out of the first sum over outputs of industries, yielding the final part of the allocative efficiency contribution (*ALLOC*), that attributable to output taxes (final line). The replacement of quantities by effective quantities and technical efficiencies in the final RHS expression yields the technical efficiency contribution (*TECH*) and the **non-opt**imising/non-price taking behaviour contribution (*NONOPT*).

Summary of welfare decomposition

Figure 1 provides summary of the welfare decomposition equations is now provided. Table 1 lists the status, in a GEMPACK implementation, of each of the items introduced in the derivation of the welfare decomposition. It also provides brief descriptions of each item.

Figure 1 **Summary of the welfare decomposition**

$$
\begin{aligned}
q^{NNP} &= TOT + ASS\_PRI \\
&\quad + RORF + FENDW \\
&\quad + ENDW + ALLOC + TECH + NONOPT \\
\tilde{o} &= \partial \ln U\left(\mathbf{P}^{C}, \mathbf{P}^{G}, P^{S}, V^{NNP}\right) \big/ \partial \ln V^{NNP} \\
TOT &= \tilde{o}.\left[V^{X}.p^{X} - V^{M}.p^{M}\right] \big/ V^{NNP} \\
ASS\_PRI &= \tilde{o}.\left[\left(V^{\acute{E}}.p^{\acute{E}} + \sum_{\ddot{o}} V_{\ddot{o}}^{FA}.\eth_{\ddot{o}}^{FA}\right) - \left(V^{S}.p^{S} + \sum_{k} V_{k}^{DEP}.\eth_{k}^{DK}\right)\right] \big/ V^{NNP} \\
RORF &= \tilde{o}.\sum_{\ddot{o}} V_{\ddot{o}}^{FA}.\tilde{n}_{\ddot{o}}^{FA} \big/ V^{NNP} \\
FENDW &= \tilde{o}.\sum_{\ddot{o}} V_{\ddot{o}}^{FA}.q_{\ddot{o}}^{FA} \big/ V^{NNP} \\
ENDW &= \tilde{o}.\left[\sum_{e,j} \hat{V}_{ej}^{\mathrm{I}}.q_{ej}^{\mathrm{I}} - \sum_{k} V_{k}^{DEP}.\left(d_{k}^{DK} + q_{k}^{DK}\right)\right] \big/ V^{NNP} \\
ALLOC &= \tilde{o}.\left[\sum_{c,f} R_{cf}^{\mathrm{I}}.q_{cf}^{\mathrm{I}} + \sum_{i,j} R_{ij}^{\mathrm{I}}.q_{ij}^{\mathrm{I}} + \sum_{c,j} R_{cj}^{\mathbf{0}}.q_{cj}^{\mathbf{0}}\right] \big/ V^{NNP} \\
TECH &= \tilde{o}.\left[\sum_{c,j} \hat{V}_{cj}^{\mathbf{0}}.a_{cj}^{\mathbf{0}} + \sum_{i,j} V_{ij}^{\mathrm{I}}.a_{ij}^{\mathrm{I}}\right] \big/ V^{NNP} \\
NONOPT &= \tilde{o}.\left[\sum_{c,j} \hat{V}_{cj}^{\mathbf{0}}.\bar{q}_{cj}^{\mathbf{0}} - \sum_{i,j} V_{ij}^{\mathrm{I}}.\bar{q}_{ij}^{\mathrm{I}}\right] \big/ V^{NNP}
\end{aligned}
$$

Table 1 **Welfare decomposition terms with guide to GEMPACK implementation**

Symbol	*Description*	*Status in GEMPACK implementation*
q^{NNP}	Percentage change in real net national product	Percentage change variable
υ	Elasticity of utility with respect to nominal net national product	Coefficient (non_parameter)
TOT	Terms of trade welfare contribution	Change variable
ASS_PRI	Asset price welfare contribution	Change variable
RORF	Foreign rate of return welfare contribution	Change variable
FENDW	Foreign endowment welfare contribution	Change variable
ENDW	Domestic endowment welfare contribution	Change variable
ALLOC	Allocative efficiency welfare contribution	Change variable
TECH	Technical efficiency welfare contribution	Change variable
NONOPT	Non-optimising and/or non-price taking producer behaviour welfare contribution	Change variable

3 Two theorems used to derive and simplify the welfare decomposition

The NNP price index

Theorem: If nominal NNP is allocated between private and government consumption goods and saving so as to maximise an utility function, then, with real NNP and the NNP price index defined as in section 2:

$$\left\{V^{NNP}.p^{NNP} - \left(V^{C}.p^{C} + V^{G}.p^{G} + V^{S}.p^{S}\right)\right\} + V^{NNP}.q^{NNP} = \frac{V^{NNP}}{\partial \ln U / \partial \ln V^{NNP}}.q^{NNP}$$

Proof: The linearisation of:

$$Q^{NNP} = U\left(\mathbf{P}^{C}, \mathbf{P}^{G}, P^{S}, V^{NNP}\right)$$

is:

$$\begin{aligned}\Delta Q^{NNP} &= \partial U/\partial \mathbf{P}^{C}.\Delta\mathbf{P}^{C} + \partial U/\partial \mathbf{P}^{G}.\Delta\mathbf{P}^{G} + \partial U/\partial P^{S}.\Delta P^{S} + \partial U/\partial V^{NNP}.\Delta V^{NNP}\\ &= \frac{\partial U}{\partial V^{NNP}}.\left(\Delta V^{NNP} - \mathbf{Q}^{C}.\Delta\mathbf{P}^{C} - \mathbf{Q}^{G}.\Delta\mathbf{P}^{G} - Q^{S}.\Delta P^{S}\right)\end{aligned}$$

where Roy's identity has been used to derive the last line. If changes are converted to percentage changes then:

$$Q^{NNP}.q^{NNP} = \frac{\partial U}{\partial V^{NNP}}.\left(V^{NNP}.v^{NNP} - \mathbf{V}^{C}.\mathbf{p}^{C} - \mathbf{V}^{G}.\mathbf{p}^{G} - V^{S}.p^{S}\right)$$

On rearranging and replacing the vector expressions with macro aggregates:

$$\begin{aligned}\left(Q^{NNP} - \frac{\partial U}{\partial V^{NNP}}.V^{NNP}\right)q^{NNP} &= \frac{\partial U}{\partial V^{NNP}}.\left(V^{NNP}.p^{NNP} - \mathbf{V}^{C}.\mathbf{p}^{C} - \mathbf{V}^{G}.\mathbf{p}^{G} - V^{S}.p^{S}\right)\\ &= \frac{\partial U}{\partial V^{NNP}}.\left(V^{NNP}.p^{NNP} - V^{C}.p^{C} - V^{G}.p^{G} - V^{S}.p^{S}\right)\end{aligned}$$

Consequently,

$$\begin{aligned}V^{NNP}.p^{NNP} - V^{C}.p^{C} - V^{G}.p^{G} - V^{S}.p^{S} &= \left(\frac{Q^{NNP}}{\partial U/\partial V^{NNP}} - V^{NNP}\right)q^{NNP}\\ &= \left(\frac{1}{\partial \ln U/\partial \ln V^{NNP}} - 1\right)V^{NNP}.q^{NNP}\end{aligned}$$

and the result follows.

QED.

The non-optimising/non-price taking behaviour effect

Theorem: If industry *j* maximises revenue, minimises costs, is a price taker and has zero pure profits then

$$\sum_{c,j} \hat{V}_{cj}^{0}.\bar{q}_{cj}^{0} = \sum_{i,j} V_{ij}^{1}.\bar{q}_{ij}^{1}$$

If these conditions are satisfied for all industries then *NONOPT*=0.

Proof: Define revenue and cost functions for industry *j*, thus:

$$\begin{aligned}
\mathrm{R}\left(\overline{\mathbf{P}}_j^{0}, \overline{\mathbf{Q}}_j^{1}\right) &= \max\left\{\overline{\mathbf{P}}_j^{0}.\overline{\mathbf{Q}} : \quad \overline{\mathbf{Q}} \in \mathrm{Y}\left(\overline{\mathbf{Q}}_j^{1}\right)\right\} \\
\mathrm{C}\left(\overline{\mathbf{P}}_j^{1}, \overline{\mathbf{Q}}_j^{0}\right) &= \min\left\{\overline{\mathbf{P}}_j^{1}.\overline{\mathbf{Q}} : \quad \overline{\mathbf{Q}} \in \mathrm{X}\left(\overline{\mathbf{Q}}_j^{0}\right)\right\}
\end{aligned}$$

where $\mathrm{Y}(\cdot)$ and $\mathrm{X}(\cdot)$ denote production possibility and inputs requirement sets, respectively. Note that the revenue and cost functions are expressed in terms of effective prices and effective quantities, which can change due to changes in prices, quantities or technical efficiencies. The derivation to be presented would not be valid if the revenue and cost functions were expressed in terms of actual prices and quantities, as these functions would implicitly embody an assumption of constant technical efficiency (that is, constant production technology).[10] The zero pure profits condition can be written in one of four ways:

$$\begin{aligned}
0 &= \mathrm{R}\left(\overline{\mathbf{P}}_j^{0}, \overline{\mathbf{Q}}_j^{1}\right) - \overline{\mathbf{P}}_j^{1}.\overline{\mathbf{Q}}_j^{1} \\
&= \overline{\mathbf{P}}_j^{0}.\overline{\mathbf{Q}}_j^{0} - \mathrm{C}\left(\overline{\mathbf{P}}_j^{1}, \overline{\mathbf{Q}}_j^{0}\right) \\
&= \mathrm{R}\left(\overline{\mathbf{P}}_j^{0}, \overline{\mathbf{Q}}_j^{1}\right) - \mathrm{C}\left(\overline{\mathbf{P}}_j^{1}, \overline{\mathbf{Q}}_j^{0}\right) \\
&= \overline{\mathbf{P}}_j^{0}.\overline{\mathbf{Q}}_j^{0} - \overline{\mathbf{P}}_j^{1}.\overline{\mathbf{Q}}_j^{1}
\end{aligned}$$

Partial differentiation, with respect to quantities, of each of the first two expressions yields:

$$\begin{aligned}
\partial \mathrm{R} \big/ \partial \overline{\mathbf{Q}}_j^{1} &= \overline{\mathbf{P}}_j^{1} \\
\partial \mathrm{C} \big/ \partial \overline{\mathbf{Q}}_j^{0} &= \overline{\mathbf{P}}_j^{0}
\end{aligned}$$

[10] The current treatment embodies the assumption that production technology only changes via changes in the technical efficiencies associated with industry outputs and inputs.

The assumption of price taking ensures that partial derivatives of prices with respect to quantities do not occur in the previous two equations, that is:

$$\partial \overline{\mathbf{P}}_j^1 \big/ \partial \overline{\mathbf{Q}}_j^1 = 0$$
$$\partial \overline{\mathbf{P}}_j^0 \big/ \partial \overline{\mathbf{Q}}_j^0 = 0$$

The assumptions of revenue maximising and cost minismising behaviour allow us to apply Shepherd's lemma to the revenue and cost functions, yielding, respectively:

$$\partial \mathrm{R} \big/ \partial \overline{\mathbf{P}}_j^0 = \overline{\mathbf{Q}}_j^0$$
$$\partial \mathrm{C} \big/ \partial \overline{\mathbf{P}}_j^1 = \overline{\mathbf{Q}}_j^1$$

Therefore, linearisation of the third expression for zero pure profits yields:

$$\begin{aligned}
0 &= \Delta \mathrm{R} - \Delta \mathrm{C} \\
&= \partial \mathrm{R} \big/ \partial \overline{\mathbf{Q}}_j^1 .\Delta \overline{\mathbf{Q}}_j^1 + \partial \mathrm{R} \big/ \partial \overline{\mathbf{P}}_j^0 .\Delta \overline{\mathbf{P}}_j^0 - \partial \mathrm{C} \big/ \partial \overline{\mathbf{Q}}_j^0 .\Delta \overline{\mathbf{Q}}_j^0 - \partial \mathrm{C} \big/ \partial \overline{\mathbf{P}}_j^1 .\Delta \overline{\mathbf{P}}_j^1 \\
&= \overline{\mathbf{P}}_j^1 .\Delta \overline{\mathbf{Q}}_j^1 + \overline{\mathbf{Q}}_j^0 .\Delta \overline{\mathbf{P}}_j^0 - \overline{\mathbf{P}}_j^0 .\Delta \overline{\mathbf{Q}}_j^0 - \overline{\mathbf{Q}}_j^1 .\Delta \overline{\mathbf{P}}_j^1 \\
&= \left[\mathbf{V}_j^1 .\overline{\mathbf{q}}_j^1 + \hat{\mathbf{V}}_j^0 .\overline{\mathbf{p}}_j^0 - \hat{\mathbf{V}}_j^0 .\overline{\mathbf{q}}_j^0 - \mathbf{V}_j^1 .\overline{\mathbf{p}}_j^1 \right] \big/ 100
\end{aligned}$$

Linearisation of the fourth expression for zero pure profits yields:

$$\begin{aligned}
0 &= \overline{\mathbf{P}}_j^0 .\Delta \overline{\mathbf{Q}}_j^0 - \overline{\mathbf{P}}_j^1 .\Delta \overline{\mathbf{Q}}_j^1 + \overline{\mathbf{Q}}_j^0 .\Delta \overline{\mathbf{P}}_j^0 - \overline{\mathbf{Q}}_j^1 .\Delta \overline{\mathbf{P}}_j^1 \\
&= \left[\hat{\mathbf{V}}_j^0 .\overline{\mathbf{q}}_j^0 - \mathbf{V}_j^1 .\overline{\mathbf{q}}_j^1 + \hat{\mathbf{V}}_j^0 .\overline{\mathbf{p}}_j^0 - \mathbf{V}_j^1 .\overline{\mathbf{p}}_j^1 \right] \big/ 100
\end{aligned}$$

The difference of the final lines in the previous two equations is equal to:

$$\begin{aligned}
0 &= \left[\mathbf{V}_j^1 .\overline{\mathbf{q}}_j^1 + \hat{\mathbf{V}}_j^0 .\overline{\mathbf{p}}_j^0 - \hat{\mathbf{V}}_j^0 .\overline{\mathbf{q}}_j^0 - \mathbf{V}_j^1 .\overline{\mathbf{p}}_j^1 \right] \big/ 100 \\
&\quad - \left[\hat{\mathbf{V}}_j^0 .\overline{\mathbf{q}}_j^0 - \mathbf{V}_j^1 .\overline{\mathbf{q}}_j^1 + \hat{\mathbf{V}}_j^0 .\overline{\mathbf{p}}_j^0 - \mathbf{V}_j^1 .\overline{\mathbf{p}}_j^1 \right] \big/ 100 \\
&= 2. \left[\mathbf{V}_j^1 .\overline{\mathbf{q}}_j^1 - \hat{\mathbf{V}}_j^0 .\overline{\mathbf{q}}_j^0 \right] \big/ 100
\end{aligned}$$

so, consequently:

$$\mathbf{V}_j^1 .\overline{\mathbf{q}}_j^1 = \hat{\mathbf{V}}_j^0 .\overline{\mathbf{q}}_j^0$$

which, when written in summation notation, is:

$$\sum\nolimits_{c,j} \hat{V}_{cj}^0 .\overline{q}_{cj}^0 = \sum\nolimits_{i,j} V_{ij}^1 .\overline{q}_{ij}^1$$

QED.

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