Heterogeneous Capital and Fixed Investment Shares in Multi-Sector Dynamic CGE Modeling[1]

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Abstract

This paper is concerned with heterogeneous capital and exogenously fixed investment shares in dynamic, multi-sector computable general equilibrium (CGE) models. In particular, it addresses the question of whether policy results of dynamic CGE models are robust with respect to investment aggregation, which is commonly done in CGE modeling. Two models – a reference model with heterogeneous capital and optimal investment and a second model with fixed investment shares and with one capital aggregate – build the framework of the analysis and offer two main conclusions. First, steady state policy effects are qualitatively identical. Second, transition paths depend on the respective aggregation rule, thus revealing that the short run impact of differential policy differs among frameworks used in dynamic CGE modeling.

Key words: *heterogeneous capital, multi-sector computable general equilibrium, capital aggregation, overlapping generations*

JEL Classification: *C68, D11, D58, D91, H31*

1. Introduction

This paper is concerned with heterogeneous capital and fixed investment shares in dynamic, multi-sector computable general equilibrium (CGE) models. It sheds light on investment (capital) aggregation rules commonly used in dynamic CGE modeling. In particular, it addresses the question of whether policy results are robust with respect to investment aggregation compared with the policy effects shown in a fully micro-founded multi-sector GE model with heterogeneous capital goods.

A close look from the point of view of a theorist on the different specifications of capital composites used in dynamic, multi-sector CGE models awakens two concerns. First, why can't we find the equivalent of a sectorally fully disaggregated *and* micro-founded general equilibrium model – i.e. a model in which sector specific capital stocks are determined by household optimization – in the CGE literature? Second, if a dynamic CGE model incorporates some kind of capital aggregation, can we expect the policy conclusions of the CGE model to be robust with respect to specific rules of capital aggregation?

The answer to the first question may well be two-fold: theoretical and empirical. On the empirical level, it is simply the lack of data (e.g., the lack of knowledge of a capital composition matrix), which precludes the specification of a detailed capital stock dynamics on a sectoral level. Concerning theory, it is the dominance of the well articulated one-capital good theory in either the *Ramsey* (1928) - *Solow* (1956) dynastic approach or the *Allais* (1947) - *Diamond* (1965) overlapping generations (OLG) approach to intertemporal allocation problems, which hinders GE users to apply theoretically much less well-articulated GE models with *heteroge-*

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neous capital goods. Theoretical sophistication hereby concerns mathematical analysis of existence, stability and comparative dynamics of steady states. While one-capital good models are well understood in this respect, existence and stability of steady states in multi-sector, multi-capital goods model were much less intensely investigated.

The answer to the second question is much more difficult to find, nonetheless it is most important for all empirical applications of multi-sector GE theory. If it were the case that capital aggregation (qualitatively) changes policy results, the employed CGE models could easily be questioned. If, on the other hand, we came to the conclusion that policy results are robust with respect to capital aggregation, we firstly had a justification for their usage in CGE modeling and secondly we knew that we could rely on analyses established so far. This second question is looked at in this paper.

In the following sections, a multi-sector, multi-capital goods variant of the overlapping generations (OLG) model with optimal investment is developed. The model represents a reference model against which a second OLG model with one capital aggregate and with fixed investment shares is compared. The two models differ primarily with respect to investment aggregation. In the reference model investment by sector of origin is derived by household optimization whereas in the other model fixed investment shares determine investment by sector of origin. The choice of the OLG approach (against the dynastic approach) is motivated by the high popularity that the *Auerbach* and *Kotlikoff* (1987) overlapping multi-cohorts approach has gained over the past decade. Subsequently, differential policy is analytically and numerically analyzed in both models in order to answer the questions posed above.

The contribution of this paper to the existing literature is threefold. First of all, it represents a *first step* in analyzing the important question of robustness of policy analysis with respect to capital aggregation, which in no way has been addressed so far. Second, the reference model against which the fixed-investment share GE model is compared is itself an innovation. While *Galor* (1992) presents a two-sector OLG model with one investment good sector (homogeneous capital) we develop a two-sector OLG model with heterogeneous capital. This model and the effects of sector-specific taxation within it has not yet been analyzed in the existing literature. Thus, the paper provides a useful framework for policy analysis of issues whose modeling requires a multi-dimensional commodity space. Third, the reference model not only further develops multi-sector OLG theory, but also represents a microfoundation for CGE models in the tradition of dynamic input-output analysis, as is explained below in more detail.

The paper offers three main conclusions. First, both models – the reference model and the fixed investment share model – predict the same qualitative steady state effects of differential policy on the economy. Second, for a wide range of parameter values, both models predict quantitatively "similar" policy results. However, for extreme parameter combinations quantitative steady state effects may differ between the two model types. Third, the number of dimensions and state variables of the reference model always exceeds the number of dimensions and state variables of fixed investment share models. This characteristic suggests differences in the transition paths that the two models generate. The analysis below especially demonstrates differences of transition paths of relative prices in the short run. Furthermore the analysis shows that overshooting regularly occurs in the reference model, which is never possible in the second model with fixed investment shares. The bottom line of the analysis is that the investment aggregation type model (with fixed investment shares) can safely replace the reference model (with investment by sector of origin derived by household optimization) for steady state policy analysis. Nevertheless both models predict different transition paths of major economic variables.

The paper is organized as follows. Section 2 is concerned with the algebraic specification and steady state analysis of a fully micro-founded two-sector OLG model with two heterogeneous capital stocks (reference model). Section 3 is devoted to the presentation of the structure and the results of the steady-state analysis of a two-sector OLG version of multi-sector dynamic CGE models with a capital aggregate and with fixed investment. Both models are then used to assess the respective steady state effects of a sector-specific policy shock in section 4. The structural differences of the equilibrium dynamics among both models and the differences in the transition paths of main variables are discussed in section 5. Finally, section 6 brings about the conclusions of the paper.

2. The Reference Model: A Two-Sector OLG Model with Heterogeneous Capital and Optimal Investments by Sector of Origin

The two-sector OLG version of the RM that is developed in this section is inspired by the seminal contribution of *Galor* (1992) and is in line with *Farmer* (1997). In *Galor*'s (1992) magisterial work on the global dynamics of a two-sector OLG model with capital accumulation the involved capital good is homogeneous since the output of only one sector is assumed to augment the capital stock (similar *Kalra* 1996). Therefore, it cannot be used as reference for FISM with a capital aggregate composed of investment goods differentiated by sector of origin. *Farmer* (1997) develops a two-sector OLG model with heterogeneous capital (investment) albeit specifies static expectations. Additionally, his analysis of the equilibrium dynamics is restricted by the assumption that the investment output of a specific sector is used as capital input in the same sector only; i.e. capital services are immobile.

The present model generalizes this approach by introducing both sector-mobile heterogeneous capital and perfect foresight. However, in the following this generalized model framework is not subjected to a rigorous analysis of the existence, uniqueness and stability of the equilibrium dynamics as in *Galor* (1992), since its main intention is the analysis of differential policy. Both the steady state effects and transitional dynamics of a simple type of tax policy will be investigated within this multi-sector OLG model with heterogeneous capital. Hence, the model provides a useful and internal consistent framework for policy analysis of issues whose modeling requires a multi-dimensional commodity space. In order to ensure that a unique and (saddle-path) stable equilibrium dynamics exists and to be in line with CGE applications the intertemporal utility function and sector production functions are algebraically specified. For analytical convenience log-linear utility and (nested) Cobb-Douglas respectively Leontief production functions are assumed.

Both the model and the policy investigated take into consideration the basic characteristics of typical models and typical policies that are dealt with in the CGE papers. These are the following:

- *consumption and aggregate savings are determined by household optimization*;
- *there are several production sectors that produce both consumption and investments goods*;
- *sectorally differentiated physical investments increase sector-of-origin specific capital stocks, which supply sector-mobile capital services*;
- *the allocation of physical investments by sector of origin among sectors of destination is governed by the equality of the rates of return on the capital goods as well as by fixed proportions of the inputs of capital goods in both sectors;*
- *the policy is sectorally differentiated*;
- *there is taxation of socially undesirable consumer behavior*;
- *revenues are recycled into the economy*.

2.1 The Reference Model

There are two industries $i = x,y$, which produce both consumption and investment goods. X_t and Y_t are the quantities produced in period t by the use of two factors, labor services

(N_t^x, N_t^y) and capital services by sector of destination D_t^x respectively D_t^y. While labor services represent original production factors, the sector-of-destination specific capital input is assumed of being a Leontief composite of sector-of-origin specific capital services. This specification is inspired by the objective to meet the above mentioned empirical problem of neoclassical growth models with heterogeneous capital goods. Additionally, the assumption of fixed-proportioned inputs of the services of sector of origin specific capital goods in RM amounts to be a natural equivalent of the assumption of fixed investment shares in FISM.

The industries operate within a fully competitive environment and maximize profits. Production is specified according to a Cobb-Douglas production function (x-sector) and following Leontief (y-sector). The growth factor of the labor force is exogenously fixed at G^L, labor productivity increases uniformly across sectors at the exogenously given factor G^τ. The natural growth factor is $G^n \equiv G^L G^\tau$. Services of x- and y-capital stocks are perfectly mobile between the production sectors implying uniform service prices Q_t^x and Q_t^y of sector-of-origin specific capital stocks. Labor is perfectly mobile at the competitively determined wage rate W_t.

The x-commodity represents the numeraire of the model. Therefore the following relative prices are defined: $p_t \equiv P_t^y / P_t^x$ $w_t \equiv W_t / P_t^x$, $q_t^i \equiv Q_t^i / P_t^x$, $1+i_{t+1} \equiv 1+r_{t+1} / P_{t+1}^x / P_t^x$, $1+r_{t+1} = Q_{t+1}^x / P_t^x = Q_{t+1}^y / P_t^y$. Relative prices are depicted as lower case letters. Additionally, we express all quantities in terms of per efficiency capita. The levels variables are transformed in the following manner: $d_t^x \equiv D_t^x / (a_t N_t^x)$, $d_t^{xx} \equiv D_t^{xx} / a_t N_t^x$, $d_t^{yx} \equiv D_t^{yx} / a_t N_t^x$, $d_t^y \equiv D_t^y / (a_t N_t^y)$, $d_t^{xy} \equiv D_t^{xy} / a_t N_t^y$, $d_t^{yy} \equiv D_t^{yy} / a_t N_t^y$, $x_t = X_t / (a_t L_t)$, $y_t \equiv Y_t / (a_t L_t)$, $l_t^x \equiv N_t^x / L_t$, $l_t^y \equiv N_t^y / L_t$, $k_t^x = K_t^x / a_t L_t$, $k_t^y = K_t^y / a_t L_t$. With this notation at hand, firm behavior can conveniently be characterized by equations (2.1) - (2.8).

$$x_t = l_t^x \left(d_t^x\right)^\alpha, \ 0 < \alpha < 1 \tag{2.1}$$

$$d_t^x = \min\left[\frac{d_t^{xx}}{b_{xx}}, \frac{d_t^{yx}}{b_{yx}}\right], 0 \le b_{xx} \le 1, 0 \le b_{yx} \le 1 \tag{2.2}$$

$$(1-\alpha)\left(d_t^x\right)^\alpha = \frac{w_t}{a_t} \tag{2.3}$$

$$\alpha\left(d_t^x\right)^{\alpha-1} = q_t^x\left(b_{xx} + b_{yx} p_{t-1}\right) \tag{2.4}$$

$$y_t = \min\left[\frac{l_t^y}{b_0}, \frac{d_t^y l_t^y}{b_1}\right], \ 0 < b_0, \ 0 < b_1 < 1 \tag{2.5}$$

$$d_t^y = \min\left[\frac{d_t^{xy}}{b_{xy}}, \frac{d_t^{yy}}{b_{yy}}\right], 0 \le b_{xy} \le 1, 0 \le b_{yy} \le 1 \tag{2.6}$$

$$p_t = b_0 \frac{w_t}{a_t} + b_1 q_t^x \left(b_{xy} + b_{yy} p_{t-1}\right) \tag{2.7}$$

$$d_t^y = \frac{b_1}{b_0}. \tag{2.8}$$

Coefficient α represents the production elasticity of capital services in *x*-production, a_t accounts for changes in labor productivity, coefficients b_0 and b_1 denote the labor and capital requirement per unit of *y*-output respectively, the coefficients $b_{ij}, i, j = x, y$ indicate the requirement of capital stock *i* services per unit of sector *j* capital input.[2]

In each period, there are two types of generations: the young household (offering labor services) and the old household (not offering labor services). Further on subscripts 1 and 2 index the typical young and old household respectively. At the beginning of period t the old household holds stocks of both commodities produced during $t-1$, denoted by K_t^x / L_{t-1} and K_t^Y / L_{t-1}. The expenditures of (per capita) *x*- and *y*-consumption are financed by the (per capita) rental income $Q_t^x (K_t^X / L_{t-1}) + Q_t^y (K_t^y / L_{t-1})$ of the typical old household. Notice that capital stocks completely depreciate after one period.[3]

At the beginning of any period, the typical young household receives wage income (plus transfers) that is used to finance the per capita consumption and investment purchases of both products $c_{t,1}^x, c_{t,1}^y, I_t^x / L_t, I_t^y / L_t$ at the nominal prices P_t^x, P_t^y. For the household to demand both investment goods, the following no-arbitrage condition is to hold:

$$q_{t+1}^x = \frac{q_{t+1}^y}{p_t} \equiv 1 + i_{t+1} . \tag{2.9}$$

The budget constraints in the first and second period of the planning horizon of the young household take the following form:

$$\left(1+t^x\right) c_{t,1}^x + \frac{I_t^x}{L_t} + p_t c_{t,1}^y + p_t \frac{I_t^y}{L_t} = w_t + tr_{t,1} \tag{2.10.a}$$

$$\left(1+t^x\right) c_{t+1,2}^x + p_{t+1} c_{t+1,2}^y = q_{t+1}^x k_{t+1}^x + tr_{t+1,2}. \tag{2.10.b}$$

Variable t^x represents a tax on x-consumption. The tax program, which is analyzed below, amounts to a simple type of differential consumption taxation. To keep the tax program as simple as possible, we assume that the tax rate is uniform across generations and does not change over time (the tax is permanent). Furthermore each generation receives transfers in proportion to its tax payment respectively. Additionally, the present value of tax payments of each generation equals the present value of its respective per capita transfers $tr_{t,1}$ and $tr_{t+1,2}$ (see *Auerbach* and *Kotlikoff* 1987, 60). Thus the income effect is eliminated and the impact of the tax program is due to the substitution effect only.

$$t^x c_{t,1}^x + \frac{t^x c_{t+1,2}^x}{1+i_{t+1}} = tr_{t,1} + \frac{tr_{t+1,2}}{1+i_{t+1}} \tag{2.11}$$

To prevent a public authority from running surpluses or deficits, we assume that in each period the revenue is completely recycled to the younger and the older generation:

$$L_t t^x c_{t,1}^x + L_{t-1} t^x c_{t,2}^x = L_t tr_{t,1} + L_{t-1} tr_{t,2} \tag{2.12}$$

Investment in period *t* increases the capital stocks in period $t+1$.

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2 Observe that capital services are fully mobile. Nonetheless their allocation is governed by the coefficients $b_{i,j}$. The motivation behind this specification is twofold. First, it weakens the criticism against multi-sector neoclassical growth models. Here the allocation of capital services is not only governed by household optimization but by technical considerations too. Second, the present specification makes the RM more similar to the FISM in which investment is governed by exogenous shares.

3 The results of the paper do not change when considering other feasible depreciation rates.

$$G^n k_{t+1}^x = \frac{I_t^x}{a_t L_t}, \quad G^n k_{t+1}^y = \frac{I_t^y}{a_t L_t} \tag{2.13}$$

Individuals are identical within as well as across generations. Individuals entering the economy at time t are characterized by the following intertemporal utility function:

$$U_{t,1} = \gamma^x \ln c_{t,1}^x + \left(1-\gamma^x\right)\ln c_{t,1}^y + \beta\left[\gamma^x \ln c_{t+1,2}^x + \left(1-\gamma^x\right)\ln c_{t+1,2}^y\right]. \tag{2.14}$$

Here γ^x and $(1-\gamma^x)$ denote the constant utility elasticities of x- and y-consumption and β denotes the time preference factor of the young household. The household entering the economy in t maximizes (2.14) subject to (2.10) and - implicitly - subject to non-negativity constraints with respect to all decision variables. The log-linearity of the utility function implies the existence of an interior solution.

Because of the competitive nature of the economy, the markets for labor (2.13), both capital services (2.14 and 2.15) and both goods (2.16) clear in each period. On account of Walras' Law, the condition for $y-$market clearing is redundant.

$$l_t^x + l_t^y = 1 \tag{2.15}$$

$$b_{xx} d_t^x l_t^x + b_{xy} d_t^y l_t^y = k_t^x \tag{2.16}$$

$$b_{yx} d_t^x l_t^x + b_{yy} d_t^y l_t^y = k_t^y \tag{2.17}$$

$$x_t = c_{t,1}^x / a_t + c_{t,2}^x \Big/ \left(G^L a_t\right) + G^n k_{t+1}^x \tag{2.18}$$

2.2 Equilibrium Dynamics and Characterization of the Steady State

The equations presented above allow for the derivation of the dynamical system of the RM. The dynamical system consists of three equations of motion (2.19) – (2.21) in the variables k^x, d^x, p.[4] These are derived in appendix A, which is available from the authors upon request.

$$\begin{aligned} k_{t+1}^x = {} & \frac{\left(\varphi-\alpha\gamma^x\right)\left(k_t^x - b_{xy} d^y\right)\left(d_t^x\right)^\alpha}{G^n \varphi\left(b_{xx} d_t^x - b_{xy} d^y\right)} - \frac{(1-\alpha)\gamma^x\left(d_t^x\right)^\alpha}{(1+\beta)G^n\varphi} \\ & + \frac{\gamma^x\left[p_t - b_0(1-\alpha)\left(d_t^x\right)^\alpha\right]\left(k_t^x - b_{xx} d_t^x\right)}{b_0 G^n \varphi\left(b_{xx} d_t^x - b_{xy} d^y\right)}, \varphi \equiv 1 + t^x\left(1-\gamma^x\right) \\ d_0^x = {} & \frac{\left(b_{xy} k_0^y - b_{yy} k_0^x\right) d^y}{b_{xx} k_0^y - b_{yx} k_0^x - \Delta d^y}, \; k_0^x \text{ and } k_0^y \text{ exogenously given} \end{aligned} \tag{2.19}$$

$$\begin{aligned} d_{t+1}^x = {} & \frac{d^y G^n k_{t+1}^x\left[b_{xy} + b_{yy} p_t\right] - b_{xy} d^y \sigma\left(d_t^x\right)^\alpha}{G^n k_{t+1}^x\left[b_{xx} + b_{yx} p_t\right] + d^y \Delta G^n p_t - b_{xx}\sigma\left(d_t^x\right)^\alpha} \text{ with } \sigma \equiv \frac{(1\text{-}\alpha)\beta}{1+\beta}, \\ \Delta \equiv {} & b_{xx} b_{yy} - b_{xy} b_{yx}, \; d_0^x = \frac{\left(b_{xy} k_0^y - b_{yy} k_0^x\right) d^y}{b_{xx} k_0^y - b_{yx} k_0^x - \Delta d^y}, \; k_0^x \text{ and } k_0^y \text{ given} \end{aligned} \tag{2.20}$$

[4] Of course, the system can be solved for (transformed into) other spaces. However, solving it for d^x, p, k^x-space results in the simpliest equations of motion.

A value of $\Delta > 0$ means that both sectors primarily make use of the services of the capital stock produced in their own sector. If $\Delta < 0$, capital services of sector *i* will be used primarily in sector *j*, $i = x, y;\ j = x, y; i \neq j$.

$$p_{t+1} = (1-\alpha) b_0 \left(d_{t+1}^x\right)^{\alpha} + \alpha b_1 \left(d_{t+1}^x\right)^{\alpha-1} \left[\frac{b_{xy} + b_{yy} p_t}{b_{xx} + b_{yx} p_t}\right] \tag{2.21}$$

Observe that stability analysis[5] shows that two eigenvalues near the steady state, if any, are lower and one is larger than one in absolute value. Therefore, the dynamic system (2.19) – (2.21) is unstable (saddle-path stable) and consists of three dimensions and two state variables, k^x and d^x (k^y). Notice further that the dynamics in prices, (2.21), is due to the no-arbitrage condition (2.9) which governs the optimal allocation of household´s wealth among the heterogeneous capital goods. A model with homogeneous capital would not include (2.9). Henceforth, the dynamics in prices were to disappear.[6]

A fixed point of this dynamical system, $k_{t+1}^x = k_t^x = k^x$, $d_{t+1}^x = d_t^x = d^x$, $p_{t+1} = p_t = p$, defines a steady state of the two-sector OLG model with heterogeneous capital and consumption taxation. A notable relation between relative prices and capital intensity can be obtained by looking at (2.21) at a steady state:

$$p\left(b_{xx} + b_{yx} p\right) = \left(b_{xx} + b_{yx} p\right)(1-\alpha) b_0 \left(d^x\right)^{\alpha} + \left(b_{xy} + b_{yy} p\right) \alpha b_1 \left(d^x\right)^{\alpha-1}. \tag{2.22}$$

This equation can be explicitly solved for p. However, it is more instructive to take the total derivative of (2.22) with respect to d^x:

$$\frac{dp}{d\left(d^x\right)} = \frac{\alpha(1-\alpha)\left(d^x\right)^{\alpha-1} p b_0 \left[b_{xx} + b_{yx} p - \left(b_{xy} + b_{yy} p\right)\left(d^y / d^x\right)\right]}{b_{yx} p^2 + (1-\alpha) b_0 b_{xx} \left(d^x\right)^{\alpha} + \alpha b_1 b_{xy} \left(d^x\right)^{\alpha-1}}. \tag{2.23}$$

Derivative (2.23) clearly shows that in a steady state the relative $y-$price decreases with rising d^x if $\left(b_{xx} + b_{yx} p\right) d^x < \left(b_{xy} + b_{yy} p\right) d^y$ and increases otherwise. These relationships can be termed the *generalized capital intensity condition* of our two-sector GE model with heterogeneous capital and optimal sectoral investments. It displays that the relative price of $y-$output decreases with rising d^x if the $x-$sector is less capital intensive than the $y-$sector, i.e. if less composite capital services per labor is required in $x-$ than in $y-$sector.

We characterize the *steady state* relations by compacting the dynamical system into two equations in k^x and d^x. Solving the utility maximization problem (2.14) subject to (2.10) and taking into account (2.11) - (2.12) yield the optimal wealth accumulation equation (A.3) in appendix A. This equation at the steady state together with (2.21) results in a locus of $k^x - d^x -$combinations (2.24), for which the wealth of young generations remains stationary. This relation is depicted as *ww*-locus in Figure 1.

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[5] The dynamic response of the reference model to a policy shock is described in section 5. Some algebraic elements of stability analysis can be found in appendix A and more numerical details are available from the authors upon request.

[6] Compare section 6 of *Galor* (1992) in which savings are independent of the real return to capital (as in our model). *Galor* (1992, 1379-1380) shows in that section that in an OLG model with homogeneous capital there is only a static relationship among the relative price and the capital intensity on the intertemporal equilibrium path.

$$k^x = \frac{(\sigma/G^n)A(d^x)^\alpha - pd^y\Delta d^x}{B}, \text{ with } A \equiv b_{xx}d^x - b_{xy}d^y, \qquad (2.24)$$
$$B \equiv (b_{xx} + pb_{yx})d^x - (b_{xy} + pb_{yy})d^y$$

(2.24) immediately displays that the *ww*-locus exhibits a pole at $B = 0$. The *ww*-locus becomes discontinuous at that d^x a case we would like to exclude. Therefore, we restrict the domain on which the *ww*-locus is defined either to the interval $(0, \bar{d}^x)$ in Figure 1.a or to the interval $(\underline{d}^x, \infty)$ in Figure 1.b.

Since k^x is required to be non-negative in steady state, both the numerator and the denumerator on the right-hand side of (2.24) have to be either positive or negative. If $B < 0$, then we encounter typically a *ww*-locus as in Figure 1.a. Otherwise, the *ww*-locus in Figure 1.b applies.

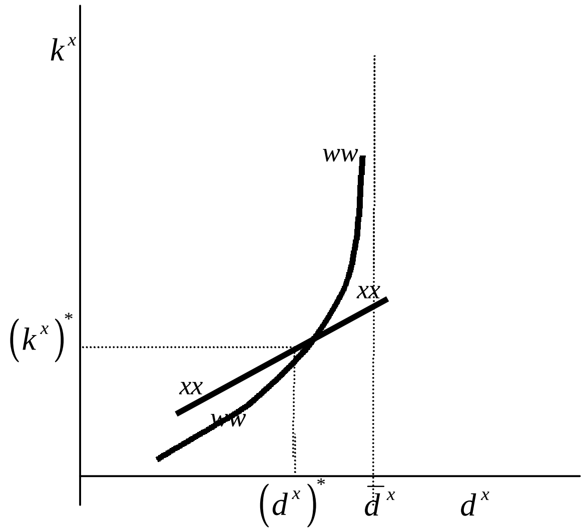

Figure 1.a

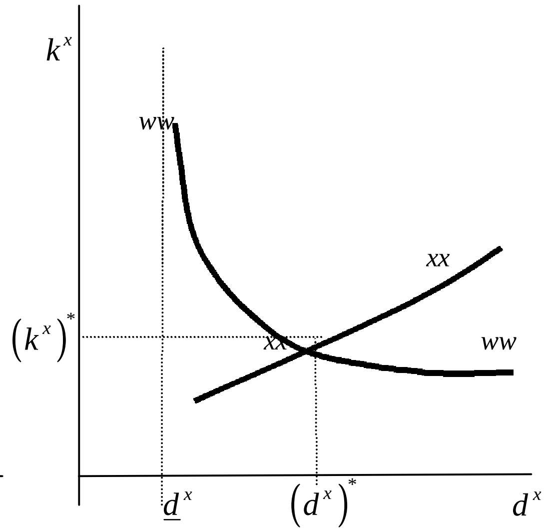

Figure 1.b

The *ww*-locus in both figures are drawn for typical numerical parameter values. However, the positive (negative) slope of the *ww*-locus in Figure 1.a (Figure 1.b) accords well to economic intuition. Consider any point on the *ww*-line in Figure 1.a (Figure 1.b). Let d^x rise, holding for the moment k^x unchanged. This means that we leave the *ww*-locus to the right. To come back on the locus, k^x has to increase (decrease) in Figure 1.a (Figure 1.b). The key to understand the relationship among d^x and k^x is offered by the equation of wealth accumulation (A.3) in steady state. If by assumption d^x rises and k^x remains unchanged, the right hand side of this equation rises because the wage rate (life time income) increases, while on the left hand side the relative price unambiguously decreases (increases) if – as in Figure 1.a (Figure 1.b) - $B < 0$ ($B > 0$). The reaction of k^y in response of the increase of d^x depends on the magnitude and the sign of Δ. If $\Delta \approx 0$[7], k^y remains approximately unchanged when d^x rises

[7] If Δ differs much from zero, there are considerable changes of k^y against rising d^x depending on the sign of Δ and A. In addition, the relationship among k^y and k^x depends non-linearly on d^x. In sum, several counteractive effects make a qualitative explanation of the slope of the $ww-$ line (as in text) impossible. However, numerical testing of a large set of admissible parameter combinations with Δ much differing from zero corroborate the qualitative reasoning in the text.

(see A.2). $\Delta \approx 0$ furthermore implies that k^x is positively proportional to k^y. Thus, before k^x changes the left hand side of the wealth accumulation equation is less (larger) than the right hand side in Figure 1.a (Figure 1.b). Consequently, k^x (and k^y) unambiguously had to rise (decrease) in order to close the negative (positive) gap between the right and the left hand side of the wealth accumulation equation. Intuitively spoken, if a larger d^x is accompanied by a smaller relative *y*-price (Figure 1.a), the larger life-time income is invested in more *y*- and more *x*-capital.

The second $k^x - d^x -$locus follows directly from the *x*-sector market clearing condition (A.6). The resulting $k^x - d^x -$relation (2.26) is referred to as *xx*-locus in Figure 1.a and Figure 1.b above.

$$k^x = C^{-1}\left\{b_{xy}d^y\left(d^x\right)^\alpha + A\left[\frac{(1-\alpha)\gamma^x\left(d^x\right)^\alpha}{(1+\beta)\varphi} + \frac{\alpha\gamma^x\sigma\left(d^x\right)^{2\alpha-1}}{\varphi G^n\left(b_{xx}+b_{yx}p\right)}\right]\right\} \tag{2.26}$$

$$\text{with } \varphi \equiv 1+\left(1-\gamma^x\right)t^x, A \equiv b_{xx}d^x - b_{xy}d^y, C \equiv \left(d^x\right)^\alpha - G^n\left(b_{xx}d^x - b_{xy}d^y\right)$$

In contrast to the *ww*-line, the slope of the *xx*-locus does not depend on the capital intensity condition. In both figures 1.a and 1.b the slope is equally positive. Notice first that the term C in (2.26) cannot become negative otherwise the maximally attainable k^x value would be negative. Given a non-negative C, the bracketed term in (2.26) must be non-negative. Next, consider any point on the *xx*-locus. Let now k^x increase and hold d^x for the moment constant. By assumption $x-$investment demand rises. To determine the change of *x*-employment and *x*-supply, we have to distinguish two cases: $A > 0$ (relevant for d^x not too small) and $A < 0$ (relevant for small d^x).

In the first case l^x and thus *x*-supply raise. $x-$consumption demand remains unchanged (since d^x does not change by assumption). Although both *x*-supply and $x-$investment demand increase. In steady state the increase in supply caused by higher k^x is larger than the enlargement of investment demand, which is identical to k^x. As a consequence, aggregate *x*-consumption demand has to rise. Therefore d^x has to increase because for d^x not too small an increase makes consumption of the younger rise by more than consumption of the older generation declines.

In the second case, *x*-supply declines with rising k^x. The emerging excess demand for the *x*-good can be alleviated only by rising d^x now reducing aggregate *x*-consumption. Aggregate *x*-consumption decreases because for low d^x a rise of d^x reduces the consumption of the old generation more than the consumption of the young generation declines. Thus, in both cases, increasing k^x requires an increase of d^x in order to return to the $xx-$line.

3. An OLG-Model with Aggregate Capital and Fixed Investment Expenditure Shares

After having presented the reference model we proceed by delineating a two-sector OLG version of CGE models with exogenous determination of investment expenditures by sector of origin. The fixed investment share model represents this predominant type of dynamic, multi-sector CGE models. In the following the latter will be referred to as FISM. The differences with respect to the RM are the following:

- *investment by sector of origin is not determined by optimization calculus but by exogenously fixed investment shares;*
- *sectoral investments are aggregated to one capital composite;*
- *there is no informational requirement concerning the composition of sectoral capital stocks.*

Except for these differences, both models the RM and the FISM are identical.

3.1 Investment Aggregation and Model Closure

As the FISM considers one capital aggregate, sector-specific capital stocks do not represent objects of choice for the young generation. As a consequence, the no-arbitrage conditions (2.7) can no longer be applied. Although aggregate savings are derived by intertemporal optimization calculus, investment *by sector of origin* cannot be determined by the household optimization alone. Fixed investment shares determine investment expenditures by sector of origin, which can be seen in most dynamic, multi-sector CGE models. In the FISM these coefficients represent exogenously fixed shares $\eta^i, (i = x, y)$ of aggregate savings. Consequently, capital in this framework can be thought of as a value capital aggregate. *Farmer and Steininger* (1999, 320) describe this framework in more detail.[8] In the same way as in the RM, capital is measured in units of the x-product. Accordingly, investment by sector of origin is determined in the following way:

$$P_t^x I_t^x = \eta^x L_t v_{t+1} = \eta^x K_{t+1} \text{ or } \frac{I_t^x}{a_t L_t} = \eta^x G^n k_{t+1},\ 0 < \eta^x < 1 \tag{3.1}$$

$$P_t^y I_t^y = \eta^y L_t v_{t+1} = \eta^y K_{t+1} \text{ or } \frac{p_t I_t^y}{a_t L_t} = \eta^y G^n k_{t+1},\ \eta^x + \eta^y = 1. \tag{3.2}$$

As in RM, capital services are assumed to be fully mobile across sectors. However, there is the significant difference that capital supply in (3.3) does not represent a usual market supply variable that is physically identifiable, but it represents the real value of a fixed-coefficient commodity composite. This is clearly only an approximation to the exact but much more information demanding RM-specification of capital services markets.

$$D_t^x + D_t^y = \frac{K_t}{P_{t-1}^x} \text{ or } d_t^x l_t^x + d_t^y l_t^y = k_t \tag{3.3}$$

As a consequence, there is only one capital rental: $q_t^y = q_t^x = q_t$. This also implies that the no-arbitrage conditions (2.7) reduce to one single condition:

$$q_t = 1 + i_t . \tag{3.4}$$

While in the RM the no-arbitrage conditions determine the allocation of savings to sectoral investments, it is the fixed investment shares in the FISM that direct savings to the different sectors. Since investment demand by sector of origin is not derived by household optimization, condition (3.4) cannot be interpreted as usual no-arbitrage condition. It rather represents a closure condition in *Sen's* (1963) sense. The exogenous investment shares also provide the missing-data-link as discussed in section two. Whenever there are no reliable data on the

[8] In some CGE applications as, e.g., *Ballard* (1989) investment shares are modeled as fixed, physical shares. In other applications as, e.g., *Farmer* and *Steininger* (1999) or *Wendner* (2000) these shares are considered as fixed nominal shares. Though, given the question looked at in this paper, this distinction does not lead to a difference in the conclusions drawn from the paper. Furthermore, nominal shares make the FISM more similar to the RM, which implicitly determines physical shares according to relative prices. The FISM also adapts physical shares according to relative prices, but rather in a framework with fixed nominal shares than in one with fixed physical shares.

composition of sectoral capital stocks, fixed investment shares allow for the determination of one composite capital stock.

3.2 Equilibrium Dynamics and Characterization of the Steady State

The dynamic system of the FISM can be characterized by one equation of motion in d^x. Its derivation is shown in appendix B, which is available from the authors upon request.

$$\frac{\alpha\gamma^x\sigma d^y}{G^n\varphi}\left(d_{t+1}^x\right)^{-1}\left(d_t^x\right)^{\alpha}-\sigma\left(\frac{\gamma^x}{\beta\varphi}+\eta^x\right)d_{t+1}^x=\frac{\sigma}{G^n}\left(\frac{\alpha\gamma^x}{\varphi}-1\right)\left(d_t^x\right)^{\alpha}+\left[1-\sigma\left(\frac{\gamma^x}{\beta\varphi}\right)+\eta^x\right]d^y \quad (3.5)$$

Stability analysis shows that the FISM is asymptotically stable in a neighborhood of a steady state. Equation (3.6) implicitly defines the steady state of the model.

$$d^x\left(\varphi-\alpha\gamma^x\right)+\left(d^x\right)^{1-\alpha}G^n d^y\left(-\frac{\varphi}{\sigma}+\varphi\eta^x+\frac{\gamma^x}{\beta}\right)-\left(d^x\right)^{2-\alpha}G^n\left(\eta^x\varphi+\frac{\gamma^x}{\beta}\right)+\alpha\gamma^x d^y=0 \quad (3.6)$$

Similar to the RM, the steady state (3.6) can be characterized by the *ww*-locus and the *xx*-locus, in $d^x-\eta^x k$-space.[9] The *ww*-locus follows from *w*ealth accumulation: $G^n k=\sigma(d^x)^{\alpha}$. In order to make this locus comparable with the *ww*-locus in the RM, it is formulated in the following way.

$$\eta^x k=\frac{\eta^x\sigma}{G^n}\left(d^x\right)^{\alpha} \quad (3.7)$$

The *xx*-locus follows from *x*-market clearing plus optimum household decisions.

$$\eta^x k=\frac{k-d^y}{G^n\left(d^x-d^y\right)}\left(d^x\right)^{\alpha}-\frac{(1-\alpha)\gamma^x\left(d^x\right)^{\alpha}}{G^n(1+\beta)\varphi}-\frac{\alpha\gamma^x\sigma\left(d^x\right)^{2\alpha-1}}{\left(G^n\right)^2\varphi} \quad (3.8)$$

Similar to the RM, the *ww*-steady state line is always monotonically increasing. A higher d^x increases lifetime income and hence savings, which fosters capital accumulation. As a consequence, a larger d^x is associated with a larger *k*. So also $\eta^x k$ increases.

The *xx*-line considers *x*-market clearing. The slope of the *xx* -line is monotonically increasing. In order to see this, consider an increase in *k* for any given d^x. The increase in *k* raises l^x and thus *x*-supply for sure, while it reduces l^y and therefore *y*-supply. For given d^x, all consumption quantities remain unchanged. However investment demand in both sectors increases due to the raise of *k*. So *y*-demand grows and establishes excess demand in the *y*-sector. In order to resolve the disequilibrium, *y*-production must be increased. This is achieved by increasing l^y (i.e. by decreasing l^x). Since $l^x=(k-d^y)/(d^x-d^y)$, *y*-supply rises with higher d^x. As a result, a higher *k* is associated with a higher d^x, which explains the slope of the *xx* -locus.

Notice the perfect identity of RM and FISM steady lines under the following conditions: $b_{xx}+b_{yx}p=1=b_{xy}+b_{yy}p$, $\eta^x=A/(d^x-d^y)=b_{xy}$, $\Delta=0$. Both models are algebraically identical if these conditions hold.

[9] Clearly not both of the two variables are dynamic variables. Though for reason of comparability with RM, we characterize the steady state by the ww- and xx-loci in $d^x-\eta^x k$-space.

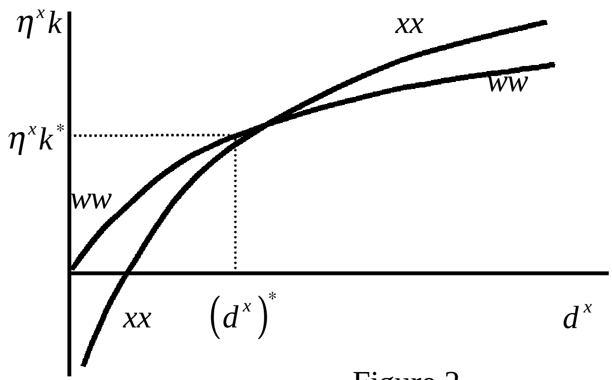

Figure 2

4. Differential Policy: Impact on the Steady State

Both preceding sections were devoted to the presentation of two alternative models, the FISM and the RM. In this and the following sections both models are employed to assess the effects of a simple type of differential policy on the economy. Consumption of the *x* commodity is taxed by a uniform and constant tax rate. The revenues are rebated to each generation in proportion to their tax payments (intertemporal income compensation). So the present value of each generations' tax payments equals their respective present value of transfers. Moreover, the government runs a balanced budget each period.[10] The impact of this policy on the steady state in each of the models is discussed and compared by means of analytical, graphical and numerical analysis in this section. The analysis shows that both models predict the same *qualitative* impact of differential policy on all economic variables *in steady state*.

4.1 Differential Policy in the FISM

The most direct approach for understanding the steady state impact of differential policy on the economy is provided by Figure 2 above. Once we know how the policy shifts the steady state lines we can infer its impact on the dynamic variable d^x. This in turn allows us to assess the effects of the policy on the remaining variables of the model.

A rise of the tax rate on *x*-consumption does not directly affect savings and thus the *ww*-line because of intertemporal income compensation. This is also shown by equation (3.7), where the tax term does not enter directly.

However optimal consumption decisions are influenced. Therefore, a rise of the tax rate on *x*-consumption makes the *xx*-line shift. The analytics of (3.8) shows whether the *xx*-line is shifted up or down as a consequence of the implementation of the tax program.

$$\frac{\partial\left[\eta^x k_{|xx}\right]}{\partial t^x} = -\frac{\eta^x\left(1-\gamma^x\right)\gamma^x\left(1-\alpha\right)dx^{\alpha-1}\left(\alpha\beta\left(d^x\right)^\alpha + d^x G^n\right)}{\left(1+\beta\right)G^n\left(\left(d^x\right)^\alpha - \left(d^x - d^y\right)\eta^x G^n\right)\varphi^2} \tag{4.1}$$

Equation (4.1) indicates the vertical shift of the *xx*-locus as a consequence of the differential tax. Since $\alpha < 1, \gamma^x < 1$ the numerator is lower than zero in any case. The sign of the denominator depends on the respective capital intensities. If the *y*-sector is more capital intensive than

[10] The objective of this specific policy program is to get rid of the income effect. The demand corresponds to Slutsky compensated demand and shows the substitution effect of the tax program only. We also analyzed another tax program, which did not eliminate income effects in a former version. The results were quite similar, however much more difficult to interpret.

the x-sector, i.e. $(d^x - d^y) < 0$, then the denominator is larger than zero. Consequently, the xx-locus shifts up as a result of taxation. Since the policy steady state is positioned on the kk-locus, both k and d^x are reduced by the tax program.

This result is easily explained economically. For given d^x and k the tax program reduces both c_1^x and c_2^x in steady state and leaves x- and y-investment unaltered. At the same time, both optimal c_1^y and c_2^y are higher. Thus, at given d^x and k there is excess supply in the x-goods market and excess demand in the y-goods markets. In order to remove the disequilibrium, the relative price must increase. The relative price simply is $p = b_0(1-\alpha)(d^x)^\alpha + b_1\alpha(d^x)^{\alpha-1}$. So $\partial p / \partial d^x = (1-\alpha)\alpha(d^x)^{\alpha-1} b_0 \left[1 - d^y / d^x\right]$. Therefore $\partial p / \partial d^x > 0$ requires d^x to decrease if $(d^x - d^y) < 0$. A decrease in d^x both increases the relative price and y-supply and thus reestablishes general equilibrium.

If $(d^x - d^y) > 0$, the numerator is still lower than zero. However the denominator is lower than zero if $(d^x)^\alpha > (d^x - d^y)\eta^x G^n$. The latter restriction always holds in a neighborhood of a steady state.[11] Thus, if $(d^x - d^y) > 0$ a marginal tax increase shifts the xx-line downward. Henceforth, k and d^x rise as a result of the tax program.

The economics behind this result resembles the argument presented before. If the x-sector is more capital intensive than the y-sector, then the relative price only rises if d^x increases because $(1 - d^y / d^x) > 0$.

The capital intensity condition determines the impact of the tax program on d^x. The impact of the tax program on the rest of the economy is straightforward. If $(d^x - d^y) < 0$ then d^x declines and the wage rate also drops. Old age x-consumption raises because of the higher rate of interest. The labor share l^x is positively correlated with d^x, so l^x drops while l^y increases. When l^y raises, y production and its relative price increase. If the relative price increases, c_1^y declines due to lower lifetime income and higher prices and c_2^y raises due to the higher rate of interest. If $(d^x - d^y) > 0$ then the impact on the supply side of the economy is the same as before because with $(d^x - d^y) > 0$ the labor share l^x is negatively correlated with d^x.

4.2 Differential Policy in the RM

The results presented above characterize the policy response according to predominant dynamic, multi-sector CGE models with policy-invariant investment shares. Now the question is addressed whether the same policy response is predicted by the RM. Notice that sectoral investment decisions are fully micro-founded (optimal) in the latter model.

Similarly as in FISM, the most direct approach for understanding the steady state impact of differential policy on the economy in RM is provided by Figures 1.a and 1.b above. Again, once we know how the policy shifts the steady state lines we can infer its impact on the dynamic variable d^x. This in turn allows us to assess the effects of the policy on the remaining variables of the model.

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[11] This can best be seen when inserting $(d^x - d^y)\eta^x G^n$ for $(d^x)^\alpha$ in the steady state equation (3.6). The resulting expression is $-\beta d^x d^y G^n - (d^x - d^y)\sigma[d^x G^n \gamma^x + \beta\eta^x(d^x(G^n - \gamma^x) + \alpha(d^x - d^y)(\gamma^x)^2)]$, which is lower than zero. Consequently, since (3.6) is equal to zero, $(d^x)^\alpha > (d^x - d^y)\eta^x G^n$.

A rise of the tax rate on *x*-consumption does not directly affect savings and thus the *ww*-line because of intertemporal income compensation. This is also shown by equation (2.24), where the tax term does not enter directly.

However optimal consumption decisions are influenced. Therefore, a rise of the tax rate on *x*-consumption makes the *xx*-line shift. The algebra of (2.25) allows to identify the cases when the *xx*-line is shifted as a consequence of the implementation of the tax program.

$$\frac{\partial k^x}{\partial t^x}\Big|_{xx} = -\frac{A\gamma^x\left(1-\gamma^x\right)}{C\varphi^2}\left[\frac{(1-\alpha)\left(d^x\right)^{\alpha}}{1+\beta}+\frac{\alpha\sigma\left(d^x\right)^{2\alpha-1}}{G^n\left(b_{xx}+b_{yx}p\right)}\right] \tag{4.2}$$

Equation (4.2) indicates the vertical shift of the *xx*-locus (for fixed d^x) as a consequence of the differential tax. The sign of the partial differential depends on the sign of A. If A is positive (relevant for d^x not too small), i. e. the *x*-sector is more *x*-capital intensive than the *y*-sector, then the differential is less than zero. Consequently, the *xx*-locus shifts down as a result of taxation. On the other hand, if $A<0$ (relevant for small steady-state d^x), the *xx*-locus shifts up. Whether A is positive or negative in the steady state solution, depends, in general, on all model parameters. However, numerical experimentation with admissible parameter sets have shown that the value of *x*-consumption share, γ^x, is most decisive for the sign of A. $\gamma^x<.5$ regularly imply steady-state d^x such that $A<0$, $\gamma^x\geq.5$ always generates steady states with $A>0$.

In both Figures 1.a and 1.b, the *xx*-line shifts down in case of $A>0$. Since the policy steady state is positioned on the *ww*-locus, in Fig. 1.a ($B<0$) with the positively sloped *ww*-locus both k^x and d^x are reduced by the tax program, while in Fig. 1.a ($B>0$) with the negatively sloped *ww*-locus, k^x decreases[12], while d^x *increases* as a consequence of *x*-consumption taxation.

The geometry of the steady-state effects of differential policy can be supplemented by the following analysis of steady-state general equilibrium policy effects. To do this, we totally differentiate the *ww*- and the *xx*-lines (2.24) and (2.25). The total differential of the *ww*-locus takes the following form:

$$dk^x-\frac{dk^x}{d\left(d^x\right)}\Big|_{ww}d\left(d^x\right)=0\,. \tag{4.3}$$

In (4.3) $dk^x\big/d\left(d^x\right)\big|_{ww}$ denotes the slope of the *ww*-locus around a steady-state solution. The total differential of the *xx*-line at a steady state reads as follows:

$$dk^x-\frac{dk^x}{d\left(d^x\right)}\Big|_{xx}d\left(d^x\right)=\frac{\partial k^x}{\partial t^x}\Big|_{xx}dt^x\,. \tag{4.4}$$

Here $dk^x\big/d\left(d^x\right)\big|_{xx}$ denotes the slope of the *xx*-locus. By means of Cramer´s rule (4.3) and (4.4) can be solved simultaneously for $dk^x\big/dt^x$ and $d\left(d^x\right)\big/dt^x$.

[12] As Figure 1.b indicates, the negative reaction of *x*-capital on taxation of *x*-consumption may be very weak. For large d^x it may be positive, indeed.

$$\frac{dk^x}{dt^x} = \frac{\frac{dk^x}{d\left(d^x\right)}\Big|_{ww} \frac{\partial k^x}{\partial t^x}\Big|_{xx}}{\frac{dk^x}{d\left(d^x\right)}\Big|_{ww} - \frac{dk^x}{d\left(d^x\right)}\Big|_{xx}} \tag{4.5}$$

$$\frac{d\left(d^x\right)}{dt^x} = \frac{\frac{\partial k^x}{\partial t^x}\Big|_{xx}}{\frac{dk^x}{d\left(d^x\right)}\Big|_{ww} - \frac{dk^x}{d\left(d^x\right)}\Big|_{xx}} \tag{4.6}$$

For a steady-state solution as in Figure 1.a (Figure 1.b), the denominators in (4.5) and (4.6) are unambiguously positive (negative). Since the numerator in (4.6) is negative (for $A > 0$), the sign of the total differential in (4.6) depends on the sign of the denominator. In (4.5) the sign of the numerator depends on the sign of the slope of the *ww*-locus: in Figure 1.a this is positive, in Figure 1.b it is negative. Thus, since an equivalent consideration is true for the denumerator in (4.5), a tax increase *always* reduces k^x, provided $A > 0$.
Finally, to provide economic intuition for these steady-state policy effects, hold for the moment d^x and k^x fixed on their pre-shock values. The tax program reduces both c_1^x and c_2^x in steady state and leaves (by assumption) *x*- and *y*-investment unaltered. At the same time, both optimal c_1^y and c_2^y go up. Thus, at given d^x and k^x (k^y) there is excess supply in the *x*-goods market and excess demand in the *y*-good market. In order to remove the disequilibrium, the relative *y*-price must increase. As known from (2.23), the relative price in steady state depends negatively (positively) on d^x when $B < 0$ ($B > 0$).Therefore, in the first case (= the situation in Figure 1.a), the higher relative price must be accompanied by a lower d^x. In the second case (= the situation in Figure 1.b), the higher relative price requires a higher d^x. The generalized capital intensity condition determines the general equilibrium impact of the tax program on d^x unequivocally.

So, the first major result of the analysis runs as follows. Both models, the RM and the FISM, predict the same qualitative steady state effects of differential policy for all economic variables.

5. The Transitional Impact of Differential Policy

The comparison of steady state effects so far did not reveal significant differences in policy effects between both model types. However, the steady state analysis captures solely long-run tendencies and has nothing to say on the transition from one steady state to another. The transition is depicted by the intertemporal equilibrium dynamics of both the RM and the FISM. Thus, before enunciating a final statement on the capability of the FISM to approximate RM, an investigation of the respective equilibrium dynamics is necessary. As we shall see, the heterogeneity of capital in the RM implies differences in the stability properties of the dynamical systems of the RM and the FISM. These differences necessitate different transition paths of major economic variables.

In particular, in RM the transition from one steady state to another occurs on a two-dimensional manifold (saddle-path) in $\left(d_t^x, k_t^x, p_t\right)$-space. As response to the tax program the relative *y*-price has to jump on the stable arm of the dynamical system in the period of implementation of the tax program. However, both capital stocks and (by means of factor market clearing conditions) the allocation of capital services across the two sectors are fixed in the period of policy implementation. So, the sudden change in the relative price represents the only possibility to respond to the policy shock directly in RM. This is a consequence of capital heterogeneity. In contrast to the response shown by the RM, the FISM reacts by adapting the allocation of capital services across sectors even in the period of policy implementation. Therefore, the initial policy responses are seen to be more moderate compared with those in the RM.

In the following section the transitional dynamics of both models is discussed and compared in some detail.

5.1 Transitional Dynamics of Differential Policy in the FISM

There is one equation of motion and one state variable, *k*, in the FISM. The latter is fixed in the period of the implementation of the tax program. However, capital services d^x and prices are free to vary. In the period of the introduction of the tax program demand for *y*-goods increases and demand for *x*-goods diminishes, while production capabilities remain the same. So the relative price must rise. If $(d^x - d^y) < 0$, d^x must decline, otherwise d^x must rise in the first policy period.

The equation of motion (3.5) shows that the transition path of d^x (and henceforth the transition paths of all variables of FISM) follow a monotonous pattern:

$$\frac{d\, d_{t+1}^x}{d\, d_t^x} = \frac{\alpha\left[\alpha\gamma^x\left(d^y - d^x\right) + \varphi d^x\right]}{\alpha\gamma^x d^y + G^n\left(\gamma^x / \beta + \eta^x\varphi\right)\left(d^x\right)^{2-\alpha}}$$

Since $\alpha\gamma^x < \varphi$, this term never changes sign as d^x rises. So d^x always follows a monotonous transition path. For that reason all other variables of FISM also adapt monotonically to the policy shock.

In a neighborhood of a steady state, the dynamical system is asymptotically stable. So it holds that

$$\left|\frac{\alpha\left[\alpha\gamma^x\left(d^y - d^x\right) + \varphi d^x\right]}{\alpha\gamma^x d^y + G^n\left(\gamma^x / \beta + \eta^x\varphi\right)\left(d^x\right)^{2-\alpha}}\right| < 1.$$

These arguments fully explain the dynamic behavior of FISM. If $(d^x - d^y) < 0$ then d^x also declines and so does the wage; the rate of interest rises. This response goes the other way round if the *x*-sector is more capital intensive than the *y*-sector. As a consequence, l^x and x^s decline, while y^s rises.

Observe two important points. First, the relative price is only *intra*temporally adapting to excess supply. This is a consequence of the homogeneity of capital. Second, the investment share η^x is exogenously fixed. Since this type of policy does not affect total savings, the investment shares also fix sectoral investment demands by origin. Both characteristics temper the policy response.

5.2 Transitional Dynamics of Differential Policy in the RM

In contrast to FISM, there are three equations of motion and two state variables in RM. By the heterogeneity of capital, the capital stocks - augmented by investments differentiated by sector of origin - are the natural candidates for the state variables. Furthermore, in RM holds the no-arbitrage condition between the rates of return on capital goods per sector of origin. Outside of a steady state, the no-arbitrage condition implies a non-trivial dynamics of relative price in RM. As already mentioned and known from the literature on descriptive growth models with heterogeneous capital of the sixties, the three-dimensional equilibrium dynamics is expected to loose the property of (local) asymptotic stability. Saddle-path ("knife-edge") stability is the best we can expect.

This saddle-path property of the RM requires the relative price to jump on a “new” saddle path in response to the tax program. In RM both capital stocks are fixed in the period of policy implementation (i.e. determined by investment decisions taken a period before), and so is the allocation of capital services across the two sectors. With d_0^x ($t=0$ is the period of policy implementation) fixed on its pre-shock value (see 2.19), the relative *y*-price in period zero must be free to jump on the stable arm of the dynamic system in RM.

To see the reason for this qualitative indeterminacy, an inspection of the formal structure of the saddle-path dynamics is useful. By means of the eigenvalues and eigenvectors to the Jacobian, the RM dynamics around a steady state can be linearly approximated as follows:

$$k_t^x = k^x + \kappa_1 (\lambda_2)^t + \kappa_2 (\lambda_3)^t, t = 0,1,2,... \tag{5.3.1}$$

$$d_t^x = d^x + \kappa_1 v_2^d (\lambda_2)^t + \kappa_2 v_3^d (\lambda_3)^t, t = 0,1,2,... \tag{5.3.2}$$

$$p_t = p + \kappa_1 v_2^p (\lambda_2)^t + \kappa_2 v_3^p (\lambda_3)^t, t = 0,1,2,..., \tag{5.3.3}$$

where $(1, v_i^d, v_i^p)'$, $i = 2,3$ represents the transpose of the normalized eigenvectors associated with the stable (less than unity) eigenvalues λ_2 and λ_3. The constants κ_j ($j = 1,2$) are determined as follows:

$$\kappa_1 = \frac{v_3^d (k_0^x - k^x) - (d_0^x - d^x)}{v_3^d - v_2^d}, \kappa_2 = \frac{(d_0^x - d^x) - v_2^d (k_0^x - k^x)}{v_3^d - v_2^d}. \tag{5.4}$$

The formal structure of the saddle-path in (d_t^x, k_t^x, p_t)-space becomes more obvious by calculating $\kappa_1 (\lambda_2)^t$ and $\kappa_2 (\lambda_3)^t$, respectively, from (5.3.1) and (5.3.2) and inserting the result into (5.3.3):

$$p_t = p + \rho^k (k_t^x - k^x) + \rho^d (d_t^x - d^x) \text{ with}$$
$$\rho^k \equiv \frac{v_2^p v_3^d - v_2^d v_3^p}{v_3^d - v_2^d}, \rho^d \equiv \frac{v_3^p - v_2^p}{v_3^d - v_2^d} \quad . \tag{5.5}$$

Two remarks on (5.5) are in order. First, since in the neighborhood of a given steady state p, k^x, d^x, ρ^k and ρ^d are constants, the saddle-path represents a linear relationship in (d_t^x, k_t^x, p_t)-space. Second, an unannounced differential tax shock in period zero changes p, k^x and d^x but does not change ρ^k and ρ^d. As a consequence, knowledge of both the steady state response of p, k^x and d^x against a tax increase and of the sign of ρ^k and ρ^d is needed to predict the qualitative nature of the price jump in period 0. However, while we have much unambiguous knowledge of the qualitative steady state response of p, k^x and d^x, it is not possible to determine algebraically the sign of ρ^k and ρ^d. We only know for sure that a taxation of *x*-consumption - irrespective of the relation between sectoral capital intensities - makes the relative *y*-price increase in new steady state. Thus, from (5.5) we are able to infer an upward (downward) jump of relative *y*-price in the period of policy implementation if

$\rho^k\left(k_0^x - k^x\right) + \rho^d\left(d_0^x - d^x\right)$ does not decrease more (less) than p rises for period 0. If both ρ^k and ρ^d were non-negative and the *y*-sector is more composite-capital intensive than the *x*-sector, an upward price jump certainly occurs. If the (new steady state) capital intensity relation is reversed and ρ^k and ρ^d remain non-negative, a downward price jump becomes more likely.

Economic intuition corroborates these analytical considerations. As explained above, the introduction of a tax on *x*-consumption cannot alter d_0^x. Consequently, the *x*-consumption of the younger household in period 0 decreases for sure while the response of *y*-consumption of the young household and *x*- and *y*-consumption of the older household also depend on the relative price jump. Let us assume for the moment that there were no change of the relative price in period 0. Then, remembering the equation of wealth accumulation for period 0, by (provisionally) given p_0, k_1^x and k_1^y cannot change. Under this proviso, the tax shock produces excess supply on x-market and excess demand on *y*-market because the *x*-consumption of the younger and the older household decrease and their *y*-consumption rise in response to the tax increase, while both *x*- and *y*-supply remain unchanged. The market imbalances must be alleviated by the alleged jump of p_0 and associated changes of k_1^x and k_1^y. Even if by construction an upward price jump seems to be unavoidable, this is not necessarily the case. The reason is that the sudden price change of p_0 and the changes of k_1^x, k_1^y and d_1^x occur simultaneously. Without knowledge of the signs of the constants and eigenvectors in (5.3.1) and (5.3.2), we are unable to say more about the qualitative nature of the price jump.

The numerical calculation of the transition paths for more than 25 sets of admissible parameter combinations in RM - which in addition lead to equivalent base case steady state solutions in FISM - show that in more than two thirds of the whole set of parameter combinations the initial price jumps upwards. Even then the response of k_1^x, k_1^y and d_1^x is mixed. While typically the initial price increase is accompanied by decreasing k_1^x and d_1^x and increasing k_1^y, also other patterns of k_1^x, k_1^y and d_1^x responses may be observed. When the initial relative price shifts down in response to differential taxation, typically k_1^x and d_1^x increase while k_1^y decrease.[13] Even here we encountered rare exceptions from this rule.

In comparison to FISM, the *initial* responses of main dynamic variables in RM to the tax shock are - not surprisingly - much more diverse and much less clear-cut. While in FISM the initial response of d^x, p and k can be unequivocally predicted from the knowledge of the capital intensity relation in (new) steady state, nothing similar is true in RM. The asymptotic instability of the dynamic system in RM and three instead of one dynamic variables preclude a qualitatively unambiguous prediction of the initial response of the dynamic variables to a differential tax shock.

However, there are not only significant differences in the initial policy responses to a tax shock among FISM and RM, but also in the subsequent transition of main dynamic variables along the transition path. Most significantly, in most cases there is no monotonous adjustment of main dynamic variables to the policy shock in RM. Intuitively this has to do with the differences of initial responses to tax policy in the two models. While in FISM by the homoge-

[13] A tax shock may result in $\Delta q_t^x > \Delta q_t^y / p_{t-1} \Leftrightarrow \alpha p_t = (b_{xx}/b_{yx}) - p_{t-1}$. The no-arbitrage condition (2.9) requires both prices to decline, which lessens the left hand side and adds to the right hand side. Since consumption shifts to the y-sector as a result of taxation, the relative price can only decline when *x*-investment grows and *y*-investment diminishes.

neity of capital d_0^x is able to respond to the policy shock, d_0^x in RM is locked in by the requirement to clear the services markets of both (historically given) heterogeneous capital stocks. Furthermore, while in FISM p_t is statically connected to d_t^x, in RM p_t is a self-contained dynamic variable; the initial value of which has also to satisfy the intertemporal no-arbitrage condition. Its forward-looking character implies that the current relative price in RM is determined by expected relative prices in the future. So the relative price in RM is by its nature much more volatile than the price in FISM. Thus, in many cases the relative price adjusts from below to the new steady state level in FISM, while adjustment takes place from above to the new steady state level in RM. In addition rigidity of d_0^x in RM lays more burden to clear the goods markets on p_0 than in FISM. So, it comes as no surprise, that in RM d_t^x and associated variables like k_t^x and k_t^y regularly "overshoot" their new long-run equilibrium levels, a phenomenon we never encounter in FISM.

6. Conclusions

The steady state effects of a differential policy can be seen to be similar under both frameworks - a model with optimal investment by sector of origin (RM) and a model with exogenously fixed investment shares (FISM). Aggregate savings decline, production, investment, the labor share and consumption drop in the *x*-sector and output, the labor share and investment rise in the *y*-sector. Both models come to the same set of conclusions concerning the comparative-dynamics of the sectoral impact of differential policy (qualitatively and to a large extent also quantitatively). However, there are three inferences concerning the transitional dynamics in which they essentially differ. All the three of them are closely related to the heterogeneity of capital in RM.

First, capital heterogeneity adds significantly to the complexity of the equilibrium dynamics, which is characterized by saddle path stability in RM and by asymptotic stability in FISM. As one consequence, in RM the dynamics of d_t^x and of closely associated variables *overshoot* the new steady state values, a phenomenon that *never* occurs in FISM type of models.

Second, as both capital stocks in RM are historically given in the period of policy implementation, there is a high burden on the relative price to adjust such that all markets clear. Transitional price reactions are more moderate in FISM where even in the first period – in contrast to RM – the sectoral allocation of capital services is free to vary.

Third, capital heterogeneity requires no arbitrage conditions to hold. These make the relative price a dynamic variable. So the predictions of RM and FISM of the initial response of the relative price to the tax policy may well be qualitatively different.

Should the FISM-type of CGE models be judged, then the following statement is in order. The FISM-type of model very closely brings forth the same policy conclusions as the RM with regard to the long run effects of differential policy. However transition paths of major economic variables in FISM-type of models do not follow the logic of fully microfounded (optimal) decisions as shown by the RM.

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