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SOLUTION TECHNIQUES FOR THE ORANI
MODEL OF THE AUSTRALIAN ECONOMY

by

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The views expressed in this paper do not necessarily reflect the opinions of the participating agencies, nor of the Australian government.
8. CONCLUDING REMARKS

The ORANI model of the Australian economy has been implemented using the method of solution described in the previous sections. The computer programs have been checked both by tests with data manufactured for 3 industries and 2 occupations (rather than 109 industries and 9 occupations), and by inspection of the 109-industry results to confirm that they are consistent with the equations of the model. The techniques described in section 7, which are desirable additions to the suite of programs but not of central importance to obtaining results, have not yet been implemented.

From a computational point of view a necessary development of the solution algorithm will be the allowance for large changes in the proportional change variables, and the associated non-linearities. One method of solution is to approximate a large change by a sequence of small changes and update the input-output table after each small change.

9. ACKNOWLEDGEMENTS

The ORANI model is basically the work of Peter Dixon, Associate Director of the IMPACT Project. The theoretical structure of the model, the linearization of the equations and the first stage of the elimination procedure are entirely his work, although he should not be held responsible for any misrepresentations of these matters in the present paper. I am grateful to Brian Parmenter for his helpful criticisms of the text of this paper.
abstract
operations in the method just described is \( 4s/a \) times the number
required for direct inversion.

Usually the part of \( A \) which is to be changed, \( H \), will
not be suitably situated near the border of \( A \). \( H \) can be moved to
the border by premultiplying and postmultiplying \( A \) by appropriate trans-
formation matrices \( K \) and \( L \) to form a new matrix \( A^* \), i.e. \( A^* = KAL \).
\( K \) and \( L \) contain only zeros except for a single 1 in each row and each
column. (The matrix multiplications are used only for the theory. Compu-
tationally there would simply be interchanges of rows and interchanges of
columns.) As \( K^{-1} = K^T \) and \( L^{-1} = L^T \), where \( T \) denotes the transpose,
the old \( A^{-1} \) can be converted to \((A^*)^{-1}\) by \((A^*)^{-1} = L^{-1}A^{-1}K^T \). Next
the new \((A^*)^{-1}\) is computed as a result of changes to \( A_{12}^* \), \( A_{21}^* \) and
\( A_{22}^* \) using the method described above. Finally, \((A^*)^{-1}\) is converted to
a new \( A^{-1} \) by \( A^{-1} = L(A^*)^{-1}K \).

7.2 Swapping variables between the endogenous and exogenous sets

Consider an ORANI solution (or part of a solution)

\[
\begin{pmatrix}
  u \\
  y
\end{pmatrix}
= 
\begin{bmatrix}
  J_{11} & J_{12} \\
  J_{21} & J_{22}
\end{bmatrix}
\begin{pmatrix}
  v \\
  z
\end{pmatrix}
\]

where \( u \) and \( y \) are vectors of endogenous variables (which may include
variables in the back solution), and \( v \) and \( z \) are vectors of exogenous variables.

\( u, v, y, z \) need not include all the relevant variables but can be restricted
to the variables of interest.
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The inverse of a matrix $A$ is denoted as $A^{-1}$ and is defined as

$$A^{-1} = \frac{1}{\text{det}(A)} \text{adj}(A)$$

where $\text{det}(A)$ is the determinant of $A$ and $\text{adj}(A)$ is the adjugate of $A$. The determinant of a matrix $A$ is given by

$$\text{det}(A) = \sum_{i=1}^{n} a_{ii} \text{det}(A_{ii})$$

for an $n \times n$ matrix $A$. The adjugate of $A$ is given by

$$\text{adj}(A) = \frac{1}{\text{det}(A)} \begin{bmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \ldots & a_{nn} \end{bmatrix}$$

for an $n \times n$ matrix $A$. The inverse of a matrix $A$ can be written as

$$A^{-1} = \text{adj}(A) / \text{det}(A)$$

where $\text{det}(A) \neq 0$. If $\text{det}(A) = 0$, then $A$ is said to be singular and does not have an inverse.

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The determinant of a matrix $A$ is a scalar value that can be computed from the elements of $A$. It is denoted by $\text{det}(A)$ and is defined as

$$\text{det}(A) = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \text{det}(A_{ij})$$

for an $n \times n$ matrix $A$, where $A_{ij}$ is the $(i,j)$-th minor of $A$ and $\text{det}(A_{ij})$ is the determinant of $A_{ij}$. The determinant of a matrix $A$ is zero if and only if $A$ is singular.

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The adjugate of a matrix $A$ is a matrix that is used to compute the inverse of $A$. The adjugate of $A$ is given by

$$\text{adj}(A) = \frac{1}{\text{det}(A)} \begin{bmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \ldots & a_{nn} \end{bmatrix}$$

for an $n \times n$ matrix $A$. The adjugate of $A$ is also the transpose of the cofactor matrix of $A$.

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The matrix inverse $A^{-1}$ of a matrix $A$ is a matrix that satisfies

$$A A^{-1} = A^{-1} A = I$$

where $I$ is the identity matrix. The matrix inverse can be computed using the formula

$$A^{-1} = \text{adj}(A) / \text{det}(A)$$

for an $n \times n$ matrix $A$. The matrix inverse exists if and only if $\text{det}(A) \neq 0$.
into column partitions of widths up to 110 columns each. This partitioning also enables the user to restrict computations to columns of interest.

The output of job (b) is a collection of matrix coefficients for a set of vector equations. The basic solution program (c) assembles these coefficients into a tableau of equations by variables, one column partition at a time. Many of the parameters of the model are input at this stage and are used to modify or form coefficients. The elimination of $q_3$ and removal of the associated equations is accomplished within the tableau. The remaining parts of the tableau are then split into matrices $A$ and $B$ according to the set of endogenous variables specified by the user.

Matrix $A$, of dimensions $250 \times 250$, is assembled and inverted as a single matrix in core. If $A$ were much larger, or if the programs were run on a computer with a smaller memory, it would be necessary to use matrix partitioning. It should be mentioned that the inversion of $A$ is performed with a standard matrix inversion package which uses the method of Gaussian elimination, including a search for maximum elements as pivots so as to minimize roundoff errors. For any meaningful economic problem, $A$ can be transformed by simple row and column transformations and multiplications to an identity matrix plus a matrix of small perturbations. It follows that the computation of the inverse is relatively insensitive to roundoff errors and that there is no need for exotic inversion techniques.

7. MODIFYING EXISTING SOLUTIONS

Once the solution to a particular problem has been computed it is often possible to efficiently derive solutions for related problems without repeating the entire solution procedure.
1. Introduction

John M. Smith

2. The OARN Model

The mathematic and computational aspects of the AIC System

3. The Report of the OARN Project

The OARN model has already been described in the previous report of the OARN Project. Volume 2.

6. Conclusions

"What is the value of interest?"

(7) Each solution is computed only for those components of the 2050 options in approximately 1250 years, which is found by means of the 9th order of approximation.
primarily with the equilibrium relationships between the different sectors of the economy, and especially the relationships between the (109) different industries. The model ignores the disequilibrium phenomena which constitute the business cycle. Instead it emphasizes underlying structural relationships. ORANI will eventually interact with a macroeconomic model, which will allow for monetary and disequilibrium phenomena, and a demographic model which will describe the size and structure of the population and workforce, including allowance for education, immigration and family formation.

The ORANI model is appropriate for examining the detailed effects of changes which may be superimposed on the economy, whether by circumstances outside Australia, e.g., by changes in world prices, or by decisions taken within Australia, e.g., by changes in tariffs, quotas, exchange rate, wage relativities, etc. For example, given a change in the import quota for motor vehicles, ORANI can be used to compute the output and employment effects in each industry, and the change in the CPI, etc., which can be attributed to this change in the quota. The model can also be used to determine which combinations of instruments could be used by the Australian government to achieve certain objectives such as a given level of real consumption or a given level of employment.

It should be emphasized that the model has been developed for the purpose of policy analysis rather than for forecasting. It is not intended to answer questions such as what will be the CPI in 4 years time. Rather, it is intended to answer questions of the kind, what will be the effect on the CPI of a specified change in the tariff rate for footwear, assuming that all other exogenous variables are held constant and assuming a specified economic environment. For ORANI the economic environment is defined by the choice of parameters, e.g., wage indexation as a percentage of the CPI, and the choice as to

However most of the variables are of no particular interest to the user. Providing there will never be a need to set them exogenously, q of the variables can be eliminated by "high school" algebra, leaving a system of \((m - q)\) equations in \((m + n - q)\) variables. The number of exogenous variables is still \(n\) but the number of endogenous variables is reduced to \((m - q)\). Solutions for the \(q\) eliminated variables can be obtained later by back substitution.

In practice the eliminations are performed in three stages,

(i) the elimination of \(q_1\) (approximately 300,000) variables which are of no further interest to the user, e.g., the value of wholesale trade associated with the flow of goods from industry \(i\) to industry \(j\);

(ii) the elimination of \(q_2\) (approximately 1,000) variables which are of interest but will always be endogenous, e.g., the prices of domestically produced goods;

(iii) the elimination of a further \(q_3\) (approximately 100) variables which are declared endogenous for the problem of interest. This elimination is performed within the solution program prior to forming \(A\) and \(B\). It was judged that the use of \(q_3 = 100\), rather than 0 or 200, minimized the total computing cost of programming effort plus computer processing.

The eliminations require specification of some of the parameters of the model. The compositions of \(q_1\) and \(q_2\) are fixed, but the composition of \(q_3\) is specified by the user. The elimination of
where $A$ is a square matrix of dimensions $m \times m$ and $B$ is of dimensions $m \times n$. This has the solution

$x = Cy$ where $C = A^{-1}B$.

This method of solution avoids any requirement for a fixed classification of variables into endogenous and exogenous. It recognizes that the model should be thought of as specifying a relationship between the endogenous variables, and that the particular classification will depend on what the user has in mind. The user has the freedom to choose which are the endogenous variables, and the remaining variables are exogenous. This option allows the same model to be used to examine quite different policies simply by altering the set of endogenous variables.

The interpretation of the elements of matrix $C$ is that a 1 per cent increase in an endogenous variable $y_1$, i.e., $\gamma = 0.01$, will produce a $C_{1j}$ per cent increase in an endogenous variable $y_j$. All other variables being held constant. In all calculations to date it has been assumed that changes are sufficiently small that the endogenous changes are proportional to the exogenous changes and that the partial contributions from different exogenous changes are additive.

The matrix multiplication of $A^{-1}B$ with $B$ are of the order of $m^2$ and $m^2n$ respectively. For the OMA model of equations, $m$ is approximately 300,000 and $n$ is approximately 1,000. Such a system is clearly too large to solve by the method just described.
3. THE EQUATIONS OF THE ORANI MODEL

The equations can be considered in three categories, (1) accounting relationships, (2) behavioural equations and (3) macroeconomic equations.

The "accounting" relationships are derived from an input-output table, that is a table which shows the flows (measured in $) of goods and services in a given year between the different sectors of the Australian economy, and in particular the flows between different industries. ORANI uses a modified version of the most recent table produced by the Australian Bureau of Statistics. This refers to the year 1968/69 and became available in 1977.

A simplified version of the ORANI input-output table is shown in figure 1. The column for each industry (or final demand category) shows the inputs into that industry. The row for each industry (or factor) shows how its output is used in other sectors of the economy. The first column shows how intermediate usage, i.e., domestic production, uses inputs of domestically produced goods, imported goods, wages, fixed capital, agricultural land and other costs. The produced goods are sold to other industries and to the final demand categories of investment, household consumption, exports and government expenditure.

The "behavioural" equations describe the choices made by the various economic agents in the model. In the production process there is the choice between imported and domestically produced inputs (assumed to be imperfect substitutes for one another), the choice as to the combination of

where \( X_1 \) and \( X_2 \) are quantities of imperfectly substitutable inputs 1 and 2 (e.g., capital and labour) which constitute the effective input \( X \), \( P_1 \) and \( P_2 \) are the unit prices of inputs 1 and 2, and \( b_1 \), \( b_2 \) and \( \rho \) are parameters.

Expressed in proportional change form, the solution is

\[
x_1 = x - \sigma (p_1 - \sum_j S_j p_j),
\]

where \( \sigma = \frac{1}{1 + \rho} \) and \( S_j \) is the share of \( P_j X_j \) in costs \( PX \). This result can be obtained directly without having to linearize a result obtained in levels\(^1\).

5. MATRIX METHOD OF SOLUTION

The complete model consists of \( m \) equations which are linear in \( m + n \) proportional change variables, where \( n > 0 \). The aim of the solution method is to solve the system of equations in a manner which is efficient with regard to computing costs and programming effort, and also allows flexibility in the choice as to which variables are endogenous and exogenous. It is also important that the user have freedom to change the values of the parameters.

The linear system of equations can be solved by specifying values for the \( n \) exogenous variables \( y \) and solving for the remaining \( m \) endogenous variables \( x \). In matrix form,

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1. See Dixon et al., op. cit., page 27.
either a budget constraint and/or a constraint on the substitution of goods, which reflects consumer preferences. Each of these choices involves a trade-off between consumption and investment. The fundamental problem is to choose a bundle of consumption goods and capital to maximize utility. Simultaneously, the consumer chooses a choice of the composition of goods. Similarly, the choice between labor and fixed capital (and educational inputs) is within the labor market.

\[
\begin{align*}
\text{Figure 1 - Input-Output Table (Partial)}
\end{align*}
\]

<table>
<thead>
<tr>
<th>Goods</th>
<th>Labor</th>
<th>Capital</th>
<th>Workers</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exports</td>
<td>I</td>
<td>I</td>
<td>I</td>
</tr>
<tr>
<td>Consumption</td>
<td>I</td>
<td>I</td>
<td>I</td>
</tr>
<tr>
<td>Investment</td>
<td>I</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Imports</td>
<td>I</td>
<td></td>
<td>I</td>
</tr>
<tr>
<td>Domestic Output</td>
<td>I</td>
<td></td>
<td>I</td>
</tr>
</tbody>
</table>

\[
\begin{align*}
\text{where terms of above are } T_x + I_x = F_x
\end{align*}
\]

The constraints on the output products in the economy are non-negative. These product equations which are linear in the production function are of the form

\[
\begin{align*}
\sum T_x + I_x = F_x
\end{align*}
\]

where \( T_x \) and \( I_x \) are the terms of the \( x \) and \( I_x \) terms in the product market. The terms on the left side are the equations of the form \( x = x + x \). Similarly, the terms on the right side are the equations of the form \( x + x = x \).

\[
\begin{align*}
\sum T_x + I_x = F_x
\end{align*}
\]

The terms on the left side are the equations of the form \( x = x + x \). Similarly, the terms on the right side are the equations of the form \( x + x = x \).
possibilities between alternative inputs. Clearly each solution depends on the relative prices of the alternative inputs.

The "macroeconomic" equations relate macroeconomic aggregates to the sums of disaggregated quantities, e.g., total employment equals the sum of employment in each occupation. They also allow for indexation of disaggregated variables to aggregate variables, e.g., the indexation of wages (by occupation) to the CPI. There are also a few equations which relate macroeconomic variables to one another. This part of the model will be expanded at a later date by addition of the macroeconomic model.

4. LINEARIZATION

Suppose that the economy can be described mathematically by the system of equations \( f(X, Y) = 0 \) where \( X \) is a vector of endogenous variables, \( Y \) is a vector of exogenous variables and \( f \) is a vector function. Initially the system is in equilibrium at state \( (X_0, Y_0) \).

If a small disturbance \( dY \) is applied, the system will eventually reach a new equilibrium state \( (X_0 + dx, Y_0 + dy) \). For sufficiently small values of \( dx \) and \( dy \), \( dy \) is given by

\[
\frac{\partial f(X_0, Y_0)}{\partial x} dx + \frac{\partial f(X_0, Y_0)}{\partial y} dy = 0 .
\] (1)

Equation (1) is linear in the increments \( dx \) and \( dy \). Consequently, providing the number of variables exceeds the number of equations, it is possible to solve for the \( n \) endogenous variables \( dx \) in terms of the \( n \) exogenous variables \( dy \) simply by solving the appropriate set of simultaneous equations. Such a form has the added advantage that it is possible to examine the effects of different policies simply by swapping variables between the two sets of variables \( dx \) and \( dy \).

Although (1) is in a linear form ORANI follows a slightly different linearization which was used by Johansen\(^1\). Equation (1) is now written

\[
\frac{\partial f(X_0, Y_0)}{\partial x} X_0 \frac{dx}{X_0} + \frac{\partial f(X_0, Y_0)}{\partial y} Y_0 \frac{dy}{Y_0} = 0 ,
\]

i.e.,

\[
Ax + By = 0 ,
\]

where

\[
A = \frac{\partial f(X_0, Y_0)}{\partial x} X_0 , \quad B = \frac{\partial f(X_0, Y_0)}{\partial y} Y_0 ,
\]

and lower case letters are now used to represent proportional changes in corresponding upper case variables, i.e.,

\[
x = \frac{dx}{X_0} \quad \text{and} \quad y = \frac{dy}{Y_0} .
\]

For the purpose of linearization the equations of the model can be considered in three groups. (These should not be confused with the three categories used in section 3.)

1. Many of the equations are of the form

\[
X_3 = W_1X_1 + W_2X_2 ,
\] (2)

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