Econometric Foundations of the Great Ratios of Economics

CoPS Working Paper No. G-300, March 2020

Don Harding
Centre of Policy Studies,
Victoria University

ISSN 1 921654 02 3 ISBN 978-1-921654-08-4

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Address: Centre of Policy Studies, Victoria University, PO Box 14428, Melbourne, Victoria, 8001
home page: www.vu.edu.au/CoPS/ email: copsinfo@vu.edu.au Telephone +61 3 9919 1877

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Don Harding[1]

March 28, 2020

[1]I thank Adrian Pagan for comments on earlier drafts, all remaining errors are mine. I acknowledge support from ARC Discovery Grant DP1801038707

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Abstract

We study the puzzle that econometric tests reject the great ratios hypothesis but economic growth theorists and quantitative macroeconomic model builders continue to embed that hypothesis in their work. We develop an econometric framework for the great ratios hypothesis and apply that framework to investigate the commonly used econometric techniques that produce rejection of the great ratios hypothesis. We prove that these methods cannot produce valid inference on the great ratios hypothesis. Thus we resolve the puzzle in favour of the growth theorists and quantitative macroeconomic model builders. We apply our framework to investigate the econometric basis for an influential paper that uses unit root and cointegration tests to reject the great ratios hypothesis for a vector that comprises consumption, financial wealth and labour income.
JEL Codes: C12, C18, C32, E00
Keywords: Great Ratios Hypothesis, Cointegration, Likelihood Ratio Inference

1 Introduction

Despite being rejected by econometric tests, the great ratios hypothesis is a favorite assumption of economic growth theorists and quantitative macroeconomic model builders. We develop an econometric framework to investigate this puzzle.

Klein and Kosobud (1961) held that certain ratios of economic variables are time invariant. In light of the evidence that many macroeconomic series are integrated of order one, I(1), the original hypothesis can be split into two hypotheses, the balanced growth hypothesis that deals with covariance stationary variables and the great ratios hypothesis that deals with integrated variables. For variables X_t and Y_t, that are both I(1) and strictly positive, this hypothesis holds that R_t^* is covariance stationary, where $R_t^* \equiv \frac{X_t}{Y_t}$. Under the alternative hypothesis R_t^* is not covariance stationary.

Since $\ln R_t^* = \ln X_t - \ln Y_t$ is also covariance stationary, inference on the great ratios hypothesis is usually made via unit root and cointegration tests. Initially, these tests failed to reject the great ratios hypothesis; see for example, Campbell (1987), King et.al. (1991), Cochrane (1994). However, on longer runs of data for the US and for the G7 countries, Harvey et. al. (2003) find that the cointegrating restrictions implied by the great ratios are rejected. This rejection is also found in the influential work of Lettau and Ludvigson (2013).

Despite the econometric evidence cited above, almost all quantitative macroeconomic models of fluctuations, whether produced by academics, central banks or commercial ventures embed some of the great ratios in their structure; see Kapetaneous et.al. (2019).

Jones (1995) placed considerable emphasis on interpreting and testing the stylized facts of growth within the unit root/cointegration framework. Twenty years later writing in *The Handbook of Macroeconomics*, Jones (2015), makes no use of these methods to interrogate, curate, interpret and analyze the facts of economic growth. In the economic growth literature avoidance of unit root and cointegration tests as a framework to collect stylized facts is clear and evidently deliberate.

After setting out Kaldor's stylized facts (which overlap with the great ratios). Jones and Romer (2010) observe that

> Redoing this exercise nearly 50 years later shows just how much progress we have made. Kaldor's first five facts have moved from research papers to textbooks. There is no longer any interesting debate about the features that a model must contain to explain them. These features are embodied in one of the great successes of growth theory in the 1950s and 1960s, the neoclassical growth model.

The puzzle that we study is that facts that are rejected on the basis of widely used econometric tests can also be described as part of the established body of knowledge by economic growth theorists. In resolving this puzzle we focus on exploring whether there is a previously unknown feature that limits the usefulness of unit root and cointegration based inference.

We work with a variable R_t that is constructed as the ratio of the numerator to the sum of numerator and denominator in the candidate great ratio, i.e. $R_t = \frac{X_t}{X_t+Y_t}$.The key insight that underpins our results is set out in section 2 where we prove that a stationary distribution always exists for R_t. Moreover, if X_t and Y_t are I(1) this stationary distribution places probability one-half on $R_t = 1$, probability one-half on $R_t = 0$ and with probability zero R_t lies in the interior of $[0, 1]$. We also fully characterize the joint and conditional distributions for (R_t, R_{t-k}).

Clearly R_t^* is a monotonic transformation of R_t, ie $R_t^* = \frac{R_t}{1-R_t}$, but the nature of the stationary distribution for R_t makes it impossible to use the stationary distribution for R_t to obtain a stationary distribution for R_t^*. Thus, our results do not contradict the well known fact that a stationary distribution cannot exist for R_t^* since it is I(1) with unbounded support.

By construction R_t is a bounded variable that lies in $[0, 1]$, we prove, in section 2, that when X_t and Y_t are I(1) and the great ratios hypothesis is false, R_t is integrated of order one in the sense used by Granger (2010) that $Corr\left(R_t, R_{t-k}\right) = 1$ for all $k > 0$. This resolves the question asked by Granger (2010) of whether a bounded process can be I(1).

To prove that there is a fundamental problem with econometric testing of the great ratios hypothesis we focus first, in section 3, on whether it is possible to produce moment based inference on that hypothesis. Then in section 4 we obtain the exact likelihood and show that its form precludes likelihood based inference.

If the great ratios hypothesis is false and the numerator and denominator in the ratio are I(1), then the stationary distribution for R_t has the property that all positive moments of R_t are equal to one-half. Thus it is natural to ask, in section 3, whether sample moments of R_t can be used to make inference on the great ratios hypothesis. We find the distribution of the sample moments and show that it precludes using them to make inference about the great ratios. This result and its explanation helps to build intuition regarding the issues that arise in making inference about the great ratios hypothesis.

We demonstrate in section 4 that unit root tests are invalid for making inference on the great ratios hypothesis. We first prove that the conditional likelihood function can be written in terms of R_t. Then we show that the stationary distribution for R_t can be used to obtain the distribution of the initial condition. Combining these we obtain the exact likelihood and show that it has properties that make inference invalid.

In section 5 we apply our framework to investigate the influential paper by Lettau and Ludvigson (2013). This paper is the ideal application for our framework since the paper uses unit root and cointegration tests to make inference about the great ratios in a three variable vector error correction model (VECM). The variables in the model are consumption, financial wealth and labour income, all in per capita terms. Thus, the findings of their paper have significant implications for the permanent income theory of consumption.

Conclusions are presented in section 6.

2 Theoretical framework

Wallis (1987) and Granger and Ding (1996) suggest that the logistic transformation can be used to map univariate variables defined on $[0, 1]$ into the real line. We use a generalization of this idea to express R_t as a continuous, monotonic transformation of a variable r_t that is defined on the real line,

$$R_t = \frac{1}{1 + g(r_t)} \tag{1}$$

where $g(.)$ is a strictly monotonically increasing, positive, continuous function and $\lim_{x \to -\infty} g(r) = -\infty$ and $\lim_{x \to \infty} g(r) = \infty$. We model r_t as the sum of a deterministic component d_t and a stochastic component z_t that is related to ε_t via the Beveridge-Nelson decomposition (3).

$$r_t = z_t + d_t \tag{2}$$

$$z_t - z_0 = \psi(1)(\varepsilon_1 + ... + \varepsilon_t) + \eta_t - \eta_0 \tag{3}$$

Where $E\varepsilon_t = 0$, $E\varepsilon_t^2 = \sigma^2$, $E\varepsilon_t\varepsilon_\tau = 0$ for $t \neq \tau$, $\sum_{j=0}^{\infty} j \left|\psi_j\right| < \infty$, $\psi(1) = \sum_{j=0}^{\infty} \psi_j$, $\delta_j = -\sum_{i=1}^{\infty} \psi_{j+i}$, $\sum_{j=0}^{\infty} \delta_j < \infty$ and $\eta_t = \sum_{j=0}^{\infty} \delta_j \varepsilon_{t-j}$ is a covariance stationary process.

There are now three cases to consider depending on whether the deterministic component, d_t, is dominated by, is of the same order or dominates $\sqrt{t}$. Theorem 1 gives the unique limiting distribution for R_t when z_t is an integrated process and the deterministic component is dominated by $\sqrt{t}$ in the sense that $\lim_{t\to\infty} \frac{d_t}{\sqrt{t}} = 0$. Later we discuss the other two cases.

Theorem 1 *Define R_t as the function of r_t in (1) where r_t is the sum of a stochastic component z_t and a deterministic component d_t via (2). The stochastic component z_t has the Beveridge-Nelson decomposition (3). The deterministic component, d_t, is assumed to have the property that $\lim_{t\to\infty} \frac{d_t}{\sqrt{t}} = 0$. The long run variance $\psi(1)$ is assumed to be non zero. Under these conditions, the limiting distribution of R_t is (4).*

$$\lim_{t\to\infty} \Pr(R_t \leq \theta) = \frac{1}{2}\ ,\ \theta \in [0,1) \tag{4}$$

And, the limiting distribution of (R_t, R_{t-k}) is (5).

$$\lim_{t\to\infty} \Pr(R_t \leq \alpha, R_{t-k} \leq \alpha) = \frac{1}{2}\ ,\ \alpha \in [0,1) \tag{5}$$

Proof. See Appendix A ■

The limiting distribution for R_t, (4), places probability one-half on $R_t = 1$, probability one-half on $R_t = 0$ and probability zero on drawing values of R_t in the interior of $[0, 1]$. The limiting distribution for (R_t, R_{t-1}) places probability one-half on $(R_t = 0, R_{t-1} = 0)$ and probability one-half on $(R_t = 1, R_{t-1} = 1)$. The limiting distribution is independent of the monotonic transformation, $g(.)$, this property guarantees the generality of our results. Importantly, it is not possible to use (1) in conjunction with (4) to obtain the stationary distribution of r_t or R_t^* ensuring that our result is consistent with received econometric theory which holds that a stationary distribution does not exist for an $I(1)$ series defined on the real line.

Corollary 1 confirms that the limiting distribution is the unique stationary distribution for R_t and also provides the the transition probabilities associated with the stationary distribution. These transition probabilities have the unusual property that the path followed by R_t is solely determined by the initial condition R_0.

Corollary 1 *The limiting distribution (4) has transition probabilities (6) and (7)*

$$\Pr\left(R_t = 1 | R_{t-k} = 1\right) = 1 \ , \forall k > 0 \tag{6}$$

$$\Pr\left(R_t = 0 | R_{t-k} = 0\right) = 1 \ , \forall k > 0. \tag{7}$$

The limiting distribution (4) is also the stationary distribution for R_t.

Proof. Let P_{ij} denote $\Pr\left(R_t = i, R_{t-1} = j\right)$, $P_{i|j}$ denote $\Pr\left(R_t = i | R_{t-k} = j\right)$ and P_i denote $\Pr\left(R_t = i\right)$ for $i, j = 0, 1$, then (6) follows from $P_{1|1} \equiv P_{11}/P_1$, (5) yields $P_{11} = \frac{1}{2}$ and (4) yields $P_1 = \frac{1}{2}$. A similar calculation yields (7). Stationarity of the distribution can be established by noting that

$$\Pr\left(R_t = j\right) = \Pr\left(R_t = j | R_{t-k} = j\right) \Pr\left(R_{t-k} = j\right) = \frac{1}{2} \qquad for \, j = 0, 1.$$

■

We now turn to the question of whether R_t is properly described as $I(1)$ or $I(0)$? This question is answered by theorem 2. The key to answering this question for a bounded process such as R_t is to focus on the correlation between R_t and R_{t-k}. As Granger (2010, p4) observed,

> the qualifying feature of an $I(1)$ process is the strong relationship between now and the distant past, so that $Corr(R_t, R_{t-k}) = 1$ for any k.

Theorem 2 *A variable R_t, that has stationary distribution (4) and transition probabilities (6) and (7) has the property that $Corr(R_t, R_{t-k}) = 1$ for all k. R_t is therefore $I(1)$.*

Proof. $Cov(R_t, R_{t-k}) = ER_tR_{t-k} - ER_tER_{t-k}$. From the stationary distribution $ER_tR_{t-k} = \Pr(R_t = 1, R_{t-k} = 1) = \frac{1}{2}$. Also $ER_t = ER_{t-k} = \frac{1}{2}$ so that $Cov(R_t, R_{t-k}) = \frac{1}{4}$. Again from the stationary distribution $Var(R_t) = Var(R_{t-k}) = \frac{1}{4}$ so $\sqrt{Var(R_t)\,Var(R_{t-k})} = \frac{1}{4}$. Thus $Corr(R_t, R_{t-k}) = \frac{1/4}{1/4} = 1$. ■

It is now convenient to discuss the two remaining cases mentioned above.

The first of these is where the deterministic component, d_t, is of the same order as $\sqrt{t}$ in the sense that $\lim_{t\to\infty} \frac{d_t}{\sqrt{t}} = b$. Where b is a constant. Here the main change is that in theorem 1 the limiting distribution becomes $\lim_{t\to\infty} \Pr(R_t \leq \theta) = q$ where q is a constant that is a function of b. Whether q is less than or greater than one-half depends on the sign of b.

The second of the remaining cases is where the deterministic component, d_t, dominates $\sqrt{t}$ in the sense that $\lim_{t\to\infty} \left|\frac{d_t}{\sqrt{t}}\right| = \infty$. Here the main change is that in theorem 1 the limiting distribution becomes degenerate in the sense that all of the probability mass is place on zero or one. Again, whether the probability mass is located on zero or one depends. on the sign of $\lim_{t\to\infty} \left|\frac{d_t}{\sqrt{t}}\right|$. Complete proofs of the propositions above have been omitted from the paper for conciseness because they all comprise straight forward variants of the proof of theorem 1.

2.1 Rewriting the great ratios hypothesis in terms of the stationary distribution for R_t

There are two main reasons for rewriting the great ratios hypothesis in terms of the stationary distribution for R_t.

First, it is more natural to express long run economic statements in terms of the stationary distribution for R_t than it is in terms of whether $\ln R_t^* - \alpha_t$ is covariance stationary. For example, consider consumption C_t and income Y_t and the ratio $R_t \equiv C_t / (C_t + Y_t)$. A stationary distribution that places probability one-half on $R_t = 0$ is saying that half the time consumption is zero. While a stationary distribution that places probability one-half on $R_t = 1$ is saying that half of the time households borrow an infinitely large amount to cover the difference between consumption and income. It is extremely unlikely that any modeler would knowingly seek to embed this assumption in their model. Yet it is exactly this assumption that is embedded into a model when it is assumed that consumption and income are not cointegrated with cointegrating vector $[1, -1]$.

Second, the literature currently contains several alternatives to the great ratios hypothesis given in the introduction. For example, Atfield and Temple (2010) study a model which allows for structural breaks, α_t, in the cointegrating equation so that $\ln R_t^* - \alpha_t$ is covariance stationary. One motivation for rewriting the great ratios hypothesis in terms of the stationary distribution for R_t is that so long as $\alpha_t/\sqrt{t}$ goes to zero such structural breaks can be included in the deterministic component d_t. This yields a unified great ratios hypothesis.

For the reasons outlined above we propose the following reformulated version of the great ratios hypothesis

H_{GR} : Strictly positive, unbounded from above, $I(1)$ variables X_t and Y_t satisfy the great ratios hypothesis if the stationary distribution for R_t, where $R_t \equiv X_t / (X_t + Y_t)$, places all of the probability mass on the interior of $[0, 1]$.

Where economic theory strongly rejects a stationary distribution for R_t that puts all of the probability mass on the endpoints of $[0, 1]$, we recommend that H_{GR} be adopted as the maintained hypothesis. In those cases where economic theory does not make a strong statement, inference must be made about H_{GR}. The remaining sections of the paper discuss the issues that arise in making such inference.

3 Positive moments of R_t exist but are useless for making inference about the great ratios

The stationary distribution (4) yields $ER_t^a = \frac{1}{2}$ for all $a > 0$. This strong result regarding the positive moments of R_t raises the question of whether inference could be made on the great ratios hypothesis using sample moments. Let $\overline{R^a}$ denote the sample moment for R_t raised to the a^{th} power.

$$\overline{R^a} \equiv \frac{1}{T}\sum_{t=1}^{T} R_t^a \tag{8}$$

Then $E\overline{R^a} = \frac{1}{2}$ and the variance of $\overline{R^a}$ is $\frac{1}{4}$ as is established in theorem 3.

Theorem 3 *Under the stationary distribution (4) and (5) the expected value of the sample mean $\overline{R^a}$ defined in (8) is one-half and the variance of that sample mean is one-quarter.*

Proof. The result for the sample mean is straightforward $E\overline{R^a} \equiv \frac{1}{T}\sum_{t=1}^{T} ER_t^a = \frac{1}{T}\sum_{t=1}^{T}\left(\frac{1}{2}\times 1 + \frac{1}{2}\times 0\right)$. The variance of the sample mean is

$$Var\left(\overline{R^a}\right) = E\left(\overline{R^a} - \frac{1}{2}\right)^2$$

Expanding the square yields

$$Var\left(\overline{R^a}\right) = E\left(\frac{1}{T}\sum_{t=1}^{T} R_t^a\right)^2 - \frac{1}{4}$$

Expanding the square of the sum yields

$$E\left(\frac{1}{T}\sum_{t=1}^{T}R_t^a\right)^2=\frac{1}{T^2}\sum_{i=1}^{T}\sum_{j=1}^{T}ER_i^aR_j^a$$

Using the stationary distribution $ER_i^aR_j^a=\Pr\left(R_t=1,R_{t-1}=1\right)\times 1^a\times 1^a+\Pr\left(R_t=0,R_{t-k}=0\right)\times 0^a\times 0^a$ which is equal to one-half. Thus

$$\frac{1}{T^2}\sum_{i=1}^{T}\sum_{j=1}^{T}ER_i^aR_j^a=\frac{1}{2}$$

yielding $Var\left(\overline{R^a}\right)=\frac{1}{4}$. ■

Although theorem 3 provides the expectation of $\overline{R^a}$ and the variance of $\overline{R^a}$ it turns out that these quantities are of no use in making inference about the great ratios hypothesis. The problem is to be found in the distribution of $\overline{R^a}$ which is established in theorem 4.

Theorem 4 *Given the stationary distribution (4) the sample mean of R_t^a has the probability distribution (9)*

$$\Pr\left(R_t^a\leq\theta\right)=\frac{1}{2}\ ,\ \theta\in[0,1) \tag{9}$$

The probability distribution (9) places zero probability on the interior of $[0,1]$. We are now in a position to understand why the sample positive moments, $\overline{R^a}$,are not useful in making inference about the great ratios hypothesis. Every test must make an assumption about the distribution of R_t under the hypothesis that the great ratios hypothesis is false. The natural choice is the stationary distribution (4). But this choice carries the implication that the alternative hypothesis is the join of two hypotheses. The first of these is that the great ratios hypothesis is false. The second is that that the process has been running for an sufficiently long time so that the sample is drawn from the stationary distribution. Because of the nature of the stationary distribution any sample with some R_t in the interior of $[0,1]$ constitutes perfectly strong evidence against the joint hypotheses just described. But, there is no logical way of separating the two hypotheses without bringing to play additional information about when the process started. Elliot and Müller (2006) provide a valuable discussion of some of the issues encountered in making such a choice of distribution for the initial condition.

4 Exact likelihood and unit roots

Starting with a conditional likelihood written in terms of r_t we make use of the fact that R_t is a monotonic transformation of r_t to rewrite that conditional likelihood in terms of R_t. We then use the fact that we know the stationary distribution for R_t to obtain an exact likelihood function.

Assume that the dynamics of r_t are well represented by a Gaussian $AR(p)$ process (10),

$$r_t = (1+\rho)\, r_{t-1} + \Delta d_t - \rho d_t + \sum_{i=1}^{p} \beta_i \Delta r_{t-i} + v_t, \qquad v_t \tilde{} N(0,\sigma) \tag{10}$$

The conditional density is

$$f(r_t|\widetilde{r}_{t-1}) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{v_t^2}{2\sigma^2}\right\} \tag{11}$$

where $\widetilde{r}_{t-1} = (r_{t-1}, \ldots, r_{t-p-1})$.

Making use of the inverse transformation,

$$r_t = g^{-1}\left(\frac{1-R_t}{R_t}\right). \tag{12}$$

Using (12) we make a change of variable to R_t so that the conditional density of R_t is

$$h\left(R_t|\widetilde{R}_{t-1}\right) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{\omega_t^2}{2\sigma^2}\right\} \frac{1}{R_t^2 g'(g^{-1}\left(\frac{1-R_t}{R_t}\right))}. \tag{13}$$

where

$$\omega_t = g^{-1}\left(\frac{1-R_t}{R_t}\right) - (1+\rho)\, g^{-1}\left(\frac{1-R_{t-1}}{R_{t-1}}\right) - \Delta d_t + \rho d_t$$
$$-\sum_{i=1}^{p}\beta_i\left(g^{-1}\left(\frac{1-R_{t-i}}{R_{t-i}}\right) - g^{-1}\left(\frac{1-R_{t-i-1}}{R_{t-i-1}}\right)\right)$$

The likelihood conditional on the first $p+1$ observations is

$$L\left(R_T, ...R_{p+2}|R_{p+1}, ..., R_1\right) \equiv \prod_{t=p+2}^{T} h\left(R_t|\widetilde{R}_{t-1}\right) \tag{14}$$

Letting $\varphi\left(R_{p+1}, ..., R_1\right)$ represent the joint density of the first $p+1$ observations, the exact likelihood function, written in terms of R_t, is

$$L\left(R_T, ..., R_1\right) = L\left(R_T, ...R_{p+2}|R_{p+1}, ..., R_1\right)\varphi\left(R_{p+1}, ..., R_1\right) \tag{15}$$

If the process has been running for a sufficiently long time then it is standard practice to use the stationary distribution for $\varphi\left(R_{p+1}, ..., R_1\right)$. Consider the case where the sample $\left(R_T, ..., R_1\right)$ lies strictly in interior of $[0,1]^T$ then $L\left(R_T, ...R_{p+2}|R_{p+1}, ..., R_1\right)$ is finite. But the probability of drawing $\left(R_{p+1}, ..., R_1\right)$ in the interior of $[0,1]^{p+1}$ is zero under the stationary distribution. Hence $\varphi\left(R_{p+1}, ..., R_1\right) = 0$. Thus the exact likelihood is zero.

In the case where some of the R_t are on the boundary $[0,1]$ then the conditional likelihood is undefined since it involves taking the logarithm of zero or unity. In this case the exact likelihood is undefined.

Clearly, these results for the exact likelihood make likelihood based inference unfeasible if the stationary distribution is used for the initial condition.

At the beginning of his Handbook of Macroeconomics chapter, Jones (2015) cites Einstein's famous quote that '[I]t is quite wrong to try founding a theory on observable magnitudes alone... It is the theory which decides what we can observe.' Most likely Jones was referring to the theory of the neoclassical growth model. Nevertheless, the insight is of considerable relevance here.

The theory set out in section 2 provides three possible cases for the sample

$(R_1, \ldots, R_T)$ that we observe:

Case 1, H_{GR} true Here the sample $(R_1, \ldots, R_T)$ lies in the interior of $[0,1]^T$ and the dynamics keep R_t in the interior of $[0,1]$.

Case 2, H_{GR} false but $R_0 \in (0,1)$. Here the sample also lies in the interior of $[0,1]^T$ but the dynamics are pushing R_t to the boundaries of zero and one.

Case 3, H_{GR} is false $R_0 = 0$ or 1 In this case the sample is a vector containing zeros or ones only.

If one observes the sample envisaged in case 1 then inference is straightforward. For samples $(R_1, \ldots, R_T)$ that lie in the interior of $[0,1]^T$ the inferential task is to distinguish between case 1 and case 2. Here the key difference between the two cases is whether the dynamics of the system are keeping R_t in the interior of $[0,1]$ rather than pushing it towards the end points. We have established in this section that the nature of the initial condition limits the effectiveness of likelihood based inference on whether there exists a unit root in the process for r_t that would push R_t to the boundary.

5 Application

In an influential paper Lettau and Ludvigson (2013) ask the question of what are the sources of fluctuations in real activity and financial markets. They address this question by studying the dynamics of three variables, consumption per capita, C_t, asset wealth per capita, A_t, and labour income per capita, Y_t.[1] They subject logarithms of the individual series to standard unit root tests and find that they are all I(1). To capture the dynamics they use a first order VECM in the logarithms of the variables.

$$\Delta \boldsymbol{x}_t = \boldsymbol{\nu} + \boldsymbol{\gamma}\boldsymbol{\alpha}' \boldsymbol{x}_{t-1} + \boldsymbol{\Gamma}\Delta \boldsymbol{x}_{t-1} + \boldsymbol{e}_t \tag{16}$$

[1] Details of the data and construction are provided in an appendix to the paper available on Martin Lettau's web page.

where $\boldsymbol{x}_t = (c_t, a_t, y_t)'$ and lower case letters denote logarithms of the level.

Lettau and Ludvigson find a single cointegrating vector $\widehat{\boldsymbol{\alpha}}' = (1, -0.18, -0.70)$. They find no other statistically significant cointegrating vector. Given the finding that all three variables are $I(1)$, the implication is that $\ln C_t - \ln Y_t$ and $\ln A_t - \ln Y_t$ are I(1) processes — that is the great ratios hypothesis is false. In previous sections we found it useful to construct the ratios $R_{CY,t}$ and $R_{AY,t}$ that are defined as follows,

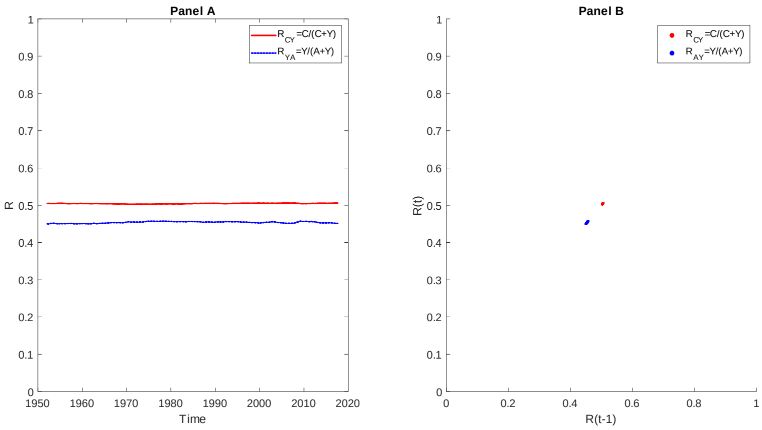
$$R_{CY,t} = \frac{C_t}{C_t + Y_t} \tag{17}$$

$$R_{AY,t} = \frac{A_t}{A_t + Y_t}. \tag{18}$$

By theorem 1, since $\ln C_t - \ln Y_t$ is I(1), with deterministic component dominated by $\sqrt{t}$, the stationary distribution for $R_{CY,t}$ places probability one-half on $R_{CY,t} = 1$, probability one-half on $R_{CY,t} = 0$ and probability zero on $R_{CY,t}$ in the interior of $[0, 1]$. The same stationary distribution also holds for $R_{AY,t}$ since $\ln A_t - \ln Y_t$ is I(1). However, the observed sample for both of these ratios lies in the interior of the $[0, 1]$ interval. Figure 1 provides some visual information regarding how far the two ratios lie inside the unit interval and whether there is evidence of dynamics that push $R_{CY,t}$ and $R_{AY,t}$ towards the boundary.

Panel A of Figure shows $R_{i,t}$ $(i = CY, AY)$, panel B of Figure plots $R_{i,t}$ against $R_{i,t-1}$. The data in panel A covers 1952Q1 to 2017Q3. In Figure the axes are chosen to extend over the $[0, 1]$ interval on which the ratios are defined so as to provide visual perspective on the task that cointegration based inference is asked to achieve. In Panel A there is little evident variation in $R_{CY,t}$ and $R_{AY,t}$ relative to the $[0, 1]$ interval on which they are defined. Despite the results from the cointegration tests on the VECM, there is no evidence in Panel A of dynamics pushing $R_{CY,t}$ and $R_{AY,t}$ towards the boundary. In Panel B the red and blue points are the data clouds for the 263 observations on $R_{CY,t}$ and $R_{AY,t}$. That these data clouds cover only a

Figure 1: Great ratios for consumption, financial wealth and labour income, United States



very small part of the $[0,1]^2$ box should serve as a warning that reliance on cointegration tests effectively means that the investigator is extrapolating outside of the domain of the sample to claim support for the hypothesis that the stationary distributions for $R_{CY,t}$ and $R_{AY,t}$ place all of the probability mass on the endpoints of $[0,1]$. There is no body of theory that can justify such extrapolation. Econometricians regularly warn students of the dangers of such extrapolation yet it is implicitly performed whenever unit root and cointegration tests are applied to make inference on the great ratios hypothesis.

5.1 Exact likelihood for VECM(1)

Here we address the question of whether the likelihood based cointegration tests referred to above suffer from the same initial value problem that we established exist for unit root tests. Again we proceed by finding the conditional and exact likelihoods for the VECM.

Assume that $\boldsymbol{e}_t$ in (16) is Gaussian with mean vector $\mathbf{0}$ and covariance matrix $\boldsymbol{\Sigma}$ then, the density of $\boldsymbol{x}_t$ conditional on $\boldsymbol{x}_{t-1}$ and $\boldsymbol{x}_{t-2}$ is

$$f\left(\boldsymbol{x}_t|\boldsymbol{x}_{t-1},\boldsymbol{x}_{t-2}\right) = (2\pi)^{-\frac{3}{2}}\left|\Sigma\right|^{-\frac{1}{2}}\exp\left\{-\frac{1}{2}e_t'\Sigma^{-1}e_t\right\}.$$

Now making use of (17) and (18) we obtain after some manipulation

$$\ln C_t = \ln \frac{R_{CY,t}}{1 - R_{CY,t}} + \ln Y_t$$

$$\ln A_t = \ln \frac{R_{AY,t}}{1 - R_{AY,t}} + \ln Y_t$$

we can now make a change of variables

$$\boldsymbol{x}_t = \begin{bmatrix} \ln \frac{R_{CY,t}}{1-R_{CY,t}} + y_t \\ \ln \frac{R_{AY,t}}{1-R_{AY,t}} + y_t \\ y_t \end{bmatrix}$$

The Jacobian J_t is

$$J_t = \begin{bmatrix} \frac{1-2R_{CY,t}}{R_{CY,t}\left(1-R_{CY,t}\right)} & 0 & 1 \\ 0 & \frac{1-2R_{AY,t}}{R_{AY,t}\left(1-R_{AY,t}\right)} & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

we can now write $\boldsymbol{\omega}_t$ in terms of the vector $\boldsymbol{R}_t$, where $\boldsymbol{R}_t = (R_{CY,t}, R_{AY,t}, y_t)$.

$$\boldsymbol{\omega}_t = \begin{bmatrix} \Delta \ln \frac{R_{CY,t}}{1-R_{CY,t}} + \Delta y_t \\ \Delta \ln \frac{R_{AY,t}}{1-R_{AY,t}} + \Delta y_t \\ \Delta y_t \end{bmatrix} - \gamma \alpha' \begin{bmatrix} \ln \frac{R_{CY,t}}{1-R_{CY,t}} + y_t \\ \ln \frac{R_{AY,t}}{1-R_{AY,t}} + y_t \\ y_t \end{bmatrix} - \Gamma \begin{bmatrix} \Delta \ln \frac{R_{CY,t-1}}{1-R_{CY,t-1}} + \Delta y_{t-1} \\ \Delta \ln \frac{R_{AY,t-1}}{1-R_{AY,t}} + \Delta y_{t-1} \\ \Delta y_{t-1} \end{bmatrix}$$

The conditional density of $\boldsymbol{R}_t$ is,

$$h\left(\boldsymbol{R}_t | \boldsymbol{R}_{t-1}, \boldsymbol{R}_{t-2}\right) = (2\pi)^{-\frac{3}{2}} |\Sigma|^{-\frac{1}{2}} \exp\left\{-\frac{1}{2}\boldsymbol{\omega}_t' \boldsymbol{\Sigma}^{-1} \boldsymbol{\omega}_t\right\} |\boldsymbol{J}_t|$$

Yielding the following conditional likelihood,

$$L\left(\boldsymbol{R}_T, ...\boldsymbol{R}_3 | \boldsymbol{R}_2, \boldsymbol{R}_1\right) = \prod_{t=p+2}^{T} h\left(\boldsymbol{R}_t | \boldsymbol{R}_{t-1}, \boldsymbol{R}_{t-2}\right)$$

Let $\lambda\left(\boldsymbol{R}_2, \boldsymbol{R}_1\right)$ be the distribution of the initial condition. Then, the exact likelihood is

$$L\left(\boldsymbol{R}_T, ..., \boldsymbol{R}_1\right) = L\left(\boldsymbol{R}_T, ...\boldsymbol{R}_3 | \boldsymbol{R}_2, \boldsymbol{R}_1\right) \lambda\left(\boldsymbol{R}_2, \boldsymbol{R}_1\right)$$

Factor $\lambda\left(\boldsymbol{R}_2, \boldsymbol{R}_1\right)$ into $\chi\left(R_{CY,2}, \ln R_{AY,t} R_{AY,2}, R_{CY,1}, R_{AY,1} | y_2, y_1\right) \pi\left(y_2, y_1\right)$ now we know that the stationary distribution for $\left(R_{CY,2}, R_{AY,2}, R_{CY,1}, R_{AY,1}\right)$ places all of the probability mass on the corners of $[0,1]^2$ thus the distribution of $\left(R_{CY,2}, R_{AY,2}, R_{CY,1}, R_{AY,1}\right)$ conditional on $\left(y_2, y_1\right)$ must equal the unconditional distribution of $\left(R_{CY,2}, R_{AY,2}, R_{CY,1}, R_{AY,1}\right)$. Since the samples in the interior of $[0,1]^2$ occur with probability zero the exact likelihood is zero for such samples. As discussed in Elliot and Müller (2006) one can try and specify a distribution other than the stationary distribution for the initial condition. But, unless there is strong information about when the system started and the nature of the initial condition such procedures are essentially ad hoc.

6 Conclusion

We have developed a formal framework for evaluating the current econometric approach to testing the great ratios. Application of that framework demonstrates that existing tests are invalid for the great ratios hypothesis.

We have proposed an alternative form of the great ratios hypothesis that is expressed in terms of the stationary distribution for a variable R_t that is constructed from the numerator and denominator in the candidate great ratio. The advantage of this alternative formulation is that it is expressed in a way that allows economic theorists and modelers to state whether or not the theory supports the hypothesis. Where theory supports the great ratios hypothesis we suggest that it should be treated as a maintained hypothesis rather than a testable hypothesis.

In cases where there is no guidance from theory unit root and cointegration tests face the problem that the exact likelihood is zero when the stationary distribution for R_t is used for the distribution of the initial condition. Inference based on the conditional likelihood or on some assumed distribution for the initial condition is ad hoc and need support from other information. We have provided an example in Figure 1 to show how visual information can help to interpret the results of unit root and cointegration tests of the great ratios hypothesis.

A Proofs and technical material

Lemmas 1 states the well known property that if $\psi(1) \neq 0$ then, $\frac{z_t - z_0}{\sqrt{t}}$ is approximately Gaussian , see Phillips and Solo (1992).

Lemma 1 *If $\psi(1) \neq 0$ then, z_t defined in (3) has the property that*

$$\Pr\left(\frac{z_t - z_0}{\sqrt{t}} < a\right) = \Phi\left(\frac{a}{\sqrt{t\sigma^2\psi^2(1)}}\right) + \kappa_t$$

where $\lim_{t\to\infty} \kappa_t = 0.$

Lemma 2 extends the result in lemma 1 to the vector $\left(\frac{z_t - z_0}{\sqrt{t}}, \frac{z_{t-k} - z_0}{\sqrt{t}}\right)'$.

Lemma 2 *If $\psi(1) \neq 0$ then, the vector $\left(\frac{z_t - z_0}{\sqrt{t}}, \frac{z_{t-k} - z_0}{\sqrt{t-k}}\right)'$ is the sum of two vectors v_t and $\boldsymbol{\xi}_t$. Where $\boldsymbol{v}_t \equiv (v_{1t}, v_{2t})'$ that is normally distributed with mean the zero vector and covariance matrix $\sum_t$*

$$\sum_t = \sigma^2 \begin{bmatrix} 1 & \sqrt{\frac{t-k}{t}} \\ \sqrt{\frac{t-k}{t}} & 1 \end{bmatrix},$$

$\lim_{t\to\infty} \boldsymbol{\xi}_t = (0,0)'$.

The proof of theorem 1 is split into two parts. Part 1 deals with the univariate limiting distribution (4) while part 2 deals with the bivariate limiting distribution (5).

Proof of theorem 1: After some manipulation of R_t in (1), using lemma 1 and the notation that $\delta_t \equiv \frac{g^{-1}\left(\frac{1-\theta}{\theta}\right) - d_t}{\sqrt{t}}$ we obtain $\Pr(R_t < \theta) =$

$1-\Pr\left(\delta_t \geq \frac{z_t - z_0}{\sqrt{t}}\right)$. Now $\lim_{t\to\infty} \delta_t = 0$ and by lemma 1 the term $\frac{z_t - z_0}{\sqrt{t}}$ converges to $N(0,1)$. The normal distribution is symmetric about zero and thus the probability that a normal random variable is less than zero is one-half, Thus $\lim_{t\to\infty} \Pr(R_t < \theta) = 1 - \frac{1}{2} = \frac{1}{2}$.

Turning to the joint density $\Pr(R_t \leq \theta, R_{t-k} \leq \theta) = 1 - \Pr\left(\delta_t \geq \frac{z_t - z_0}{\sqrt{t}}, \delta_{t-k} \geq \frac{z_{t-k} - z_0}{\sqrt{t-k}}\right)$. We can factor the joint probability as

$$\Pr\left(\delta_t \geq \frac{z_t - z_0}{\sqrt{t}}, \delta_{t-k} \geq \frac{z_{t-k} - z_0}{\sqrt{t-k}}\right) = \Pr\left(\delta_t \geq \frac{z_t - z_0}{\sqrt{t}} | \delta_{t-k} \geq \frac{z_{t-k} - z_0}{\sqrt{t-k}}\right) \times \Pr\left(\delta_{t-k} \geq \frac{z_{t-k} - z_0}{\sqrt{t-k}}\right) \quad (19)$$

Since $z_t - z_0 = z_{t-k} - z_0 + \psi(1)(\varepsilon_t + \ldots + \varepsilon_{t-k+1}) + \eta_t - \eta_{t-k+1}$ we conclude that the limit as t goes to infinity of the first term is 1 while for the second term the limit is one-half. Thus $\lim_{t\to\infty} \Pr(R_t \leq \theta, R_{t-k} \leq \theta) = \frac{1}{2}$.

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