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Sato's Insight on the Relationship between the Frisch 'Parameter' and the Average Elasticity of Substitution

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Sato's Insight on the Relationship between the Frisch 'Parameter' and the Average Elasticity of Substitution

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This short note demonstrates that Sato's 1972 insight concerning the equivalence between Frisch's 'money flexibility' parameter and the average elasticity of substitution among commodities needs to be modified if it is to be applied to non-homothetic utility functions. Fortunately the modification is easily implemented.

For those disliking the cardinalist interpretation of additive preferences, Sato's 1972 article offered an ordinalist alternative. Whereas Frisch (1959) had focussed on ω, the elasticity with respect to total expenditure of the marginal utility of an optimally spent dollar, Sato suggested that $1/\omega$ (sometimes called the *money flexibility* 'parameter') be reinterpreted simply as σ, the average elasticity of substitution. Directly additive utility functions would then provide a parametrically parsimonious demand specification quite free from the cardinalist taint.

This note has three purposes: (i) to point out that Sato's relationship

$$\sigma = -1/\omega \tag{1}$$

is not (quite) correct for non-homothetic utility functions; (ii) to define the *average* substitution elasticity precisely; and (iii) to state a relationship between σ and ω which holds for any directly additive utility function.

Point (i) is easily demonstrated by appeal to an example. Consider the case of the two-commodity linear expenditure system (LES):

$$p_i x_i = p_i \gamma_i + \beta_i (y - p_1 \gamma_1 - p_2 \gamma_2) \qquad (i=1,2), \tag{2}$$

evaluated at the following values of prices $\{p_1, p_2\}$, nominal expenditure y, and parameters:

Exogenous variable settings:
$p_1 = p_2 = 1; \qquad y = 6$.
Parameter settings:
$\beta_1 = 0.6,\ \beta_2 = 0.4;\ \gamma_1 = 0.2,\ \gamma_2 = 2.8.$

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1 The author is grateful to Maureen T. Rimmer for comments.

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Then, using a well known expression for ω we can compute $-1/\omega$ as

$$-1/\omega \;=\; \frac{y - p_1\,\gamma_1 - p_2\gamma_2}{y} \;=\; 0.5 \;, \tag{3}$$

whereas the elasticity of substitution is

$$\sigma \;=\; \frac{\beta_1(x_2 - \;\gamma_2)}{x_2} \;+\; \frac{\beta_2(x_1 - \;\gamma_1)}{x_1} \;=\; 0.54. \tag{4}$$

Since there is only one pair of commodities, no ambiguity about the definition of the *average* substitution elasticity is possible, and we find that (1) is contradicted. Choosing the homothetic special case of the LES (by setting $\gamma_1 = \gamma_2 = 0$ in the above example), we find that $-1/\omega$ does indeed equal σ (both being unity in the example).

The average substitution elasticity in the n-commodity case is now developed from the marginal-budget-share-weighted average of partial substitution elasticities σ_{ij} over all pairs involving commodity i.[3] The latter average is

$$S^i \;=\; \sum_{j\neq i} \frac{W_j\,E_j}{(1 - W_iE_i\;)}\;\sigma_{ij} \;, \qquad (i,\, j=1,\, \ldots,\, n) \tag{5}$$

where $W_i \;\equiv\; p_i\,x_i/y$ is the (average) budget share of i, and E_i is the elasticity of demand for i with respect to total expenditure. The average substitution elasticity σ is then defined to be the marginal-budget-share-weighted sum of the S^i:

$$\sigma \;=\; \sum_{i=1}^{n} W_i\,E_i\,S^i \;. \tag{6}$$

From Houthakker (1960) we know that the Allen-Uzawa partial substitution elasticities of an additive utility function are:

$$\sigma_{ij} \;=\; \phi\,E_i\,E_j \;, \qquad (i\neq j;\; i,\, j=1,\, \ldots,\, n) \tag{7}$$

where $\phi \equiv -1/\omega$. Substitution of (5) and (7) into (6) establishes

$$\sigma = \mu\,\phi = -\mu/\omega \tag{8.1}$$

where[4]

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3 This development broadly follows the appendix of Agrawal and Powell (1992); σ is defined there, however, using average (rather than marginal) budget shares as weights. Notice that the weights in (5) are normalized to sum to unity over $j \neq i$ for each $i = 1, \ldots, n$.

4 If σ is defined using average budget shares as weights (as in Agrawal and Powell (1992)), then μ in (8.2) must be replaced by χ, where χ is obtained from μ in (8.2) by reducing the power of each E term by unity.

$$\mu = \sum_{i=1}^{n} \left\{ \frac{W_i \, E_i^2}{(1 - W_i \, E_i)} \sum_{j \neq i} W_j \, E_j^2 \right\} \quad . \tag{8.2}$$

The variable μ is often close to unity in practice. It is exactly unity whenever the utility function is homothetic (that is, when $E_i = 1$ for all i). Returning to our initial example in which $E_1 = 1.8$ and $E_2 = 0.6$, we see that

$$\mu = (2/6) \times (1.8)^2 \times 0.6 + (4/6) \times (0.6)^2 \times 1.8$$

$$= 27/25 = 1.08,$$

which is the ratio of σ (0.54) to ϕ (0.5). Formulae (8.1) and (8.2) allow convenient computation of the average substitution elasticity for any directly additive utility function.

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