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A GENERALIZED INTERTEMPORAL MODEL OF COMMODITY DEMANDS AND LABOUR SUPPLY

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1. INTRODUCTION

The current specification of the IMPACT framework allows for the determination of commodity demands as a function of prices and total consumption within the ORANI model (Dixon, Parmenter, Sutton and Vincent (1982)), with labour supply decisions and other demographic influences being provided by the BACHUROO model (Sams (1979); Sams and Williams (1983)), and with the consumption/savings choice being provided (in the initial versions of the paradigm - see Powell and Lawson (1975)) by the MACRO model (Cooper and McLaren (1980)), but more recently by extended versions of ORANI (e.g., as in Meagher and Parmenter (1985)). Since recent developments in the theory of labour supply have emphasised the interrelatedness of commodity demand and leisure choices within a life-cycle context (see, for example, Heckman (1974), Ghez and Becker (1975), Abbott and Ashenfelter (1976), Heckman (1976), Wales and Woodland (1976, 1977), Ashenfelter and Ham (1979), Barnett (1979), Heckman and MaCurdy (1980), Atkinson and Stern (1981), Deaton and Muellbauer (1981), MaCurdy (1981), Pollak and Wales (1981), Blundell and Walker (1982), Killingsworth (1983), Attfield and Browning (1985), Browning, Deaton and Irish (1985) and Yatchew (1985)), there would seem to be some advantages in transferring to ORANI those decision variables within BACHUROO which, in view of the above-mentioned literature, are best modelled within a dynamic model of a joint decision-making household.

This approach would seem to be a natural extension of the recent evolution of the IMPACT paradigm. The general proposal is that the ORANI household demand system, currently based on a static allocation model, be embedded within a dynamic household decision making model. Such a framework allows for the consistent modelling not only of commodity demands, but also the demand for leisure (and hence the supply of labour) and total consumption demand. This would provide an alternative means of endogenizing the consumption/saving choice with ORANI. BACHUROO would retain its role of providing a demographic accounting framework.

In Section 2 the basic model is introduced by means of a dynamic prototype. The Extended Linear Expenditure System (ELES) and an extended version of the Almost Ideal Demand System (denoted EAIDS) are used as examples, and a General Dynamic Model which incorporates each of these as special cases is also presented. Section 3 then extends this model to allow for age-specific effects, thus providing the mechanism by which the demographic accounting of BACHUROO may specifically influence consumption, commodity demands and labour supply. This model is further extended in Section 4 to allow for the particular wage profiles that have been emphasized in the labour supply literature. Finally, Section 5 contains some concluding comments.

2. THE DYNAMIC PROTOTYPE

2.1 Theory

In order to treat commodity demands and labour supplies within a general framework, it is convenient to introduce the concept of full expenditure and to treat labour supply as the converse of the 'commodity' leisure. Thus leisure is bought back at the price of leisure which is the wage rate, and full expenditure is defined to include the initial endowments of time of the members of the household, valued at their respective wage rates.

Let q be an m vector of commodities and leisure categories, and let p be the corresponding price vector. By convention the last $m-n$ elements of q and p will refer to leisure and wage rates, respectively. Let x be full expenditure.

The allocation decisions at any point in time can be characterised by the indirect utility function:

$$g(x, p) = \max_q \{f(q) : p'q \leq x, q \geq 0\} \quad ,$$

where $f(q)$ is the direct utility function, and the 'Marshallian' system of commodity and leisure demands are derived from $g(x, p)$ by Roy's Identity

$$q = q^M (x, p) = -g_p/g_x \quad ,$$

where the M superscript indicates Marshallian demands and subscripts indicate partial differentiation.

To introduce the dynamic optimisation problem, let t stand for the time at which decisions are made and, in this prototype, suppose for expositional simplicity that the planning horizon is infinite. The intertemporal analogue of full expenditure is full wealth:

$$w = \int_t^\infty e^{-r(s-t)} p'\eta \, ds + a \quad ,$$

where a is the initial endowment of financial wealth, η is a vector of commodity and time endowments, and r is the rate of interest. It is further assumed, in the prototype, that static expectations are held with respect to the price vector p and the interest rate r.

The intertemporal decisions can be characterised by the optimal value function:

$$V(w, p) = \max \int_t^\infty e^{-\delta(s-t)} g(x(s), p) \, ds \quad ,$$

$$\text{subject to: } \dot{w}(s) = rw(s) - x(s)$$

$$w(t) = w$$

$$\lim w(s) \geq 0 \; ,$$

where the maximisation is with respect to time paths for x and w.

There are several approaches to obtaining the optimal response functions for this type of problem. For our purposes it is convenient to note that the optimal response functions satisfy the Hamilton-Jacobi equation identically:

$$(2.1) \qquad V_w \dot{w} \equiv \delta V - g .$$

All the approaches make use of the first order condition:

$$(2.2) \qquad V_w = g_x .$$

Much of the recent work in intertemporal aspects of labour supply theory makes use of the approach of Heckman (1976), Heckman and MaCurdy (1980), MaCurdy (1981), Attfield and Browning (1985) and Browning, Deaton and Irish (1985), in which the marginal optimal value of wealth, V_w, plays a key role. Let $y = V_w$. Then, since $g_x(x, p) = y$, it follows that in a strictly concave region of g, $x = \phi(y, p)$, where ϕ is g_x^{-1}. Browning, Deaton and Irish (1985) make use of the fact that 1/y may be interpreted as the price of utility to characterise preferences by the 'profit' function:

$$(2.3) \qquad h(1/y, p) = \max_u\{(1/y)u - e(u, p)\} ,$$

where u is the utility level or 'output', and e(u, p) is the expenditure (or cost) function obtained by inversion of the indirect utility function i.e. $g(e(u,p),p) \equiv u$.

Upon specification of a profit function h(1/y, p), the Frisch (or marginal utility compensated) demand functions may be obtained from Hotelling's lemma as:

$$(2.4) \qquad q = q^F(y, p) = - h_p(1/y, p) .$$

Let $h_o = \partial h(1/y, p)/\partial(1/y)$. Then h_o may be interpreted as optimal 'output' u, and the associated marginal utility compensated cost function may be reconstructed from (2.3) as:

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$$x = \phi(y, p) = (1/y)h_o - h \quad . \tag{2.5}$$

To operationalise (2.4) and (2.5), a procedure is needed to handle the unobservable variable y. Since y is the costate variable, it follows from a control theoretic formulation of the problem that:

$$\dot{y} = \delta y - H_w \quad ,$$

where H is the current-value Hamiltonian

$$H = g(x, p) + y[rw - x] \quad .$$

Hence

$$\dot{y} = (\delta - r)y \quad .$$

If there is no reason for a household to revise its plans, this smooth path in y may be used to write the Frisch demands in terms of y(o), which may then be treated as a fixed effect.

To allow for demographic surprises, and hence for revisions to the plan, it seems more appropriate to synthesise the response functions by substituting for y a function of the current state which, in the intertemporal problem considered here, may be summarised by w and p. Since $y = V_w$, this approach suggests that the optimal value function V be examined.

Partial differentiation of the Hamilton-Jacobi equation (2.1) with respect to w leads, in view of (2.2), to:

$$V_w r + V_{ww}\dot{w} = \delta V_w \quad ,$$

so that the optimal savings and full expenditure response functions are

$$\dot{w} = (\delta - r)\, V_w/V_{ww} \quad , \tag{2.6a}$$

(2.6b) $x = rw + (r - \delta) V_w/V_{ww}$.

To eliminate y from (2.4) and (2.5), it is possible to use (2.6), re-expressed in terms of y. Specifically, since $w = \psi(y, p)$, where $\psi \equiv V_w^{-1}$, the identity $w \equiv \psi(V_w(w, p), p)$ yields $1 = \psi_y V_{ww}$, so that (2.6a) and (2.6b) become

(2.7a) $\dot{w} = (\delta - r) y\psi_y$,

(2.7b) $x = r\psi + (r - \delta)y\psi_y$.

A comparison of (2.5) and (2.7b) suggests that the ϕ and ψ functions are related via the differential operator:

(2.8) $\phi = D \psi$,

where $D = rI + (r - \delta) y\partial/\partial y$.

2.2 Examples

Example 1: ELES

Consider the profit function:

(2.9) $h(1/y, p) = (1/y)[\ln(1/y) - 1 - \ln P_1] - P_o$,

where P_o and P_1 are homogeneous of degree one (HD1) functions of p. Specifically, (2.9) is the ELES profit function when $P_o = p'\gamma$ and $\ln P_1 = \Sigma_j \beta_j \ln(p_j/\beta_j)$. Although the profit function is not generally specified in characterising preferences associated with ELES, it is the natural starting point in this type of analysis. That (2.9) is in fact the ELES profit function could be demonstrated by specifying f(q) to be Klein-Rubin, constructing g(x, p), inverting g to obtain e(u, p), and finally applying (2.3) to derive h(1/y, p).

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Applying Hotelling's lemma yields the Frisch demands:

(2.10) $$q_j^F\,(y,\,p) = \partial P_o/\partial p_j + (\partial \ln P_1/\partial p_j)/y = \gamma_j + (\beta_j/p_j)/y\ ,$$

while the full expenditure equation for this example is, from (2.5):

(2.11) $$\phi(y,\,p) = P_o + 1/y\ .$$

The Marshallian version of LES would be obtained by solving (2.11) for y and substituting in (2.10). However, full expenditure is endogenous and the purpose in exhibiting (2.11) in Frisch form is to demonstrate how the intertemporal aspect of the problem is solved by using the differential operator (2.8).

Conjecturing a functional form for $w = \psi(y,\,p)$,

(2.12) $$\psi(y,\,p) = \theta_o P_o + \theta_1/y\ ,$$

use of (2.8) yields:

(2.13) $$\phi(y,\,p) = r\theta_o P_o + \delta\theta_1/y$$

and comparison with (2.11) gives the conditions $\theta_o = 1/r$, $\theta_1 = 1/\delta$. Applying these in (2.12) and solving for y as a function of w and p enables y to be eliminated from the Frisch demands (2.10) to give ELES,

(2.14) $$q_j = \gamma_j + (\beta_j/p_j)\,\delta[w - p'\gamma/r]\ .$$

Example 2: EAIDS

Consider the profit function:

(2.15) $$h(1/y,\,p) = (1/y)(1/P_2)[\ln(1/y) + \ln(1/P_2) - 1 - \ln P_1]\ ,$$

where P_1 and P_2 are HD1 and HD0, respectively. Specifically, (2.15) is the EAIDS profit function when P_1 is translog and P_2 is Cobb-Douglas.

Applying Hotelling's lemma yields the Frisch demands:

(2.16) $$q_j^F(y, p) = [\frac{\partial \ln P_1}{\partial p_j} - \frac{\partial \ln P_2}{\partial p_j} \ln(yP_2P_1)]/(yP_2) ,$$

while the full expenditure equation is, from (2.5),

(2.17) $$\phi(y, p) = 1/(yP_2) .$$

The Marshallian version of AIDS would be obtained by solving (2.17) for y and substituting in (2.16).

Proceeding as in Example 1, conjecture a functional form for $w = \psi(y, p)$,

(2.18) $$\psi(y, p) = \theta/(yP_2) .$$

Use of (2.8) yields

(2.19) $$\phi(y, p) = \delta\theta/(yP_2)$$

and comparison with (2.17) gives the condition $\theta = 1/\delta$. Applying this in (2.18) and solving for $y = 1/(\delta P_2 w)$ enables y to be eliminated from the Frisch demands (2.16) to give EAIDS,

(2.20) $$q_j = [\frac{\partial \ln P_1}{\partial p_j} + \frac{\partial \ln P_2}{\partial p_j} \ln(\frac{\delta w}{P_1})] \delta w .$$

Example 3: A General Dynamic Model

Consider the profit function:

(2.21) $$h(1/y, p) = (1/y)(1/P_2)\{[(1/y)^{-\varepsilon}(1/P_2)^{-\varepsilon}P_1^{\varepsilon} - 1]/(-\varepsilon) - 1\} - P_o ,$$

straightforward to allow a demographic accounting model to provide the weights to be applied to the demographic specific demands. Further types of demographic effects may be incorporated by the specific inclusion of demographic factors in the profit function in much the same way as these factors have been included in the indirect utility function by, for example, Pollak and Wales (1981). In addition, the model of this paper has the advantage of endogenizing the consumption/savings decision in a consistent manner.

it becomes possible to make comparisons across demographic groups which are parameterised in terms of their wage profiles. For example, this model allows for the possibility that two individuals with equal full wealth and identical tastes may demonstrate different consumption patterns. This is illustrated by the consumption function for this model:

$$x(t) = P_o(t) + [1/\theta_1(t)][w(t) - \theta_o(t)P_o(t)] ,$$

where

$$\theta_o(t) = \int_t^T e^{-\int_t^s [r(v) - \dot{P}_o(v)/P_o(v)]dv} ds ,$$

$$\theta_1(t) = \int_t^T e^{-\int_t^s \{\varepsilon[r(v) - \dot{P}_1(v)/P_1(v)] + (1-\varepsilon)[\delta + \dot{P}_2(v)/P_2(v)]\}dv} ds ,$$

$$w(t) = \int_t^T e^{-\int_t^s r(v)dv} p(s)'n ds + a(t) .$$

In a practical application, P_o, P_1, and P_2 would be chosen as specific functions of p, and the time profile of p would need to be specified. The literature suggests that the stylized facts about the wage profile could be captured by specifying p(s) quadratic in s (see Killingsworth (1983)).

5. CONCLUSION

The model outlined in this paper should be potentially quite useful within the context of computable general equilibrium models. The resulting sets of behavioural equations allow for interactions among commodity demands and labour supply, for the effect of age per se, and for the effect of wage profiles that may differ across demographic groups. It is then quite

which contains (2.9) and (2.15) as special cases.

By Hotelling's lemma the Frisch demands are:

$$(2.22) \quad q_j^F(y, p) = \partial P_o/\partial p_j + \{(\partial \ln P_1/\partial p_j)(yP_1P_2)^\varepsilon - (1-\varepsilon)(\partial \ln P_2/\partial p_j)\frac{(yP_1P_2)^\varepsilon - 1}{\varepsilon}\}/(yP_2) ,$$

while the full expenditure equation is, from (2.5),

$$(2.23) \quad \phi(y, p) = P_o + P_1{}^\varepsilon P_2{}^{\varepsilon-1} y^{\varepsilon-1} .$$

The Marshallian version of the demand system is obtained by eliminating y from (2.22) using (2.23). This yields:

$$(2.24) \quad q_j^M(x, p) = \partial P_o/\partial p_j + (\partial \ln P_1/\partial p_j)(x - P_o) - (1-\varepsilon)(\partial \ln P_2/\partial p_j)\frac{[(x-P_o)/P_1]^{\frac{\varepsilon}{\varepsilon-1}} - 1}{\varepsilon}\left(\frac{x-P_o}{P_1^\varepsilon}\right)^{\frac{1}{1-\varepsilon}} .$$

Turning now to the intertemporal component of the problem, the presence of y in the full expenditure function (2.23) may be accommodated, as in the previous examples, by conjecturing a similar form for $w = \psi(y, p)$. Specifically, consider the possibility:

$$(2.25) \quad \psi(y, p) = \theta_o P_o + \theta_1 P_1^\varepsilon P_2^{\varepsilon-1} y^{\varepsilon-1} .$$

Use of (2.8) yields:

$$(2.26) \quad \phi(y, p) = r\theta_o P_o + [\varepsilon r + (1-\varepsilon)\delta]\theta_1 P_1^\varepsilon P_2^{\varepsilon-1} y^{\varepsilon-1} ,$$

giving the conditions $\theta_o = 1/r$, $\theta_1 = 1/[\varepsilon r + (1-\varepsilon)\delta]$. Applying this in (2.25) yields a solution for y as a function of w and p, (2.27) $y = \{(w - P_o/r)[\varepsilon r + (1-\varepsilon)\delta]P_1^{-\varepsilon}\}^{\frac{1}{\varepsilon-1}}/P_2$,

which may be used in conjunction with the Frisch demands (2.22) to form a generalised intertemporal demand system.

In terms of the commodity allocation equations, the Marshallian demand functions (2.24) include as special cases LES, AIDS and possible variants such as the S-Branch. In terms of intertemporal allocation, the consumption function (full expenditure function) can be obtained from (2.23) and (2.25) as

(2.28) $x = P_o + [\varepsilon r + (1-\varepsilon)\delta](w - P_o/r)$.

3. THE DYNAMIC MODEL INCORPORATING AGE EFFECTS

3.1 Theory

By assuming an infinite time horizon, the simple model of Section 2 effectively allows decisions to be independent of the age of the decision-maker. Replacement of the infinite horizon by a finite horizon leads to a life cycle model in which the age of the decision maker has a specific influence on expenditure decisions. This type of model is well suited to the IMPACT paradigm. If the systems of demand equations which appear within ORANI are parameterised in terms of the stage of the life cycle, then a demographic accounting model such as the relevant module in BACHUROO, can be used to weight ORANI's age specific demand system to provide an aggregate demand system with age specific demographic effects.

Let the typical life-time be T. Then with a slight variation in notation the optimal value function evaluated at any specific age t, where $0 \leq t \leq T$, may be defined as:

incorporating age specific wages and prices are best illustrated by an example.

4.2 An Example

Continuing with the example of the previous sections, let g_x^{-1} be defined by:

(4.1) $\phi(y, p) = P_o + P_1^{\varepsilon}P_2^{\varepsilon-1}y^{\varepsilon-1}$.

For $w = \psi(y, \pi, t)$, consider the form:

(4.2) $\psi(y, \pi, t) = \theta_o(t)P_o(t) + \theta_1(t)P_1(t)^{\varepsilon}P_2(t)^{\varepsilon-1}y^{\varepsilon-1}$.

Applying the operator (3.5) yields:

$$D\psi = [(r - \dot{P}_o/P_o)\theta_o - \dot{\theta}_o]P_o$$
$$+ \{[\varepsilon(r-\dot{P}_1/P_1) + (1-\varepsilon)(\delta + \dot{P}_2/P_2)]\theta_1 - \dot{\theta}_1\}P_1^{\varepsilon}P_2^{\varepsilon-1}y^{\varepsilon-1} ,$$
$$= (\lambda_o\theta_o - \dot{\theta}_o)P_o + (\lambda_1\theta_1 - \dot{\theta}_1)P_1^{\varepsilon}P_2^{\varepsilon-1}y^{\varepsilon-1}, \text{ say,}$$

and equality of coefficients between ϕ and $D\psi$ means that θ_o and θ_1 must satisfy $\dot{\theta}_o = \lambda_o\theta_o - 1$, $\dot{\theta}_1 = \lambda_1\theta_1 - 1$. These conditions are satisfied by setting

$$\theta_i(t) = \int_t^T e^{-\int_t^s \lambda_i(v)dv} ds , \quad i = 0, 1 ,$$

where the terminal condition $w(T) = 0$ has been used to eliminate the constant of integration.

The importance of this formulation over and above that of the previous section is that it specifically allows for an anticipated wage profile. Thus

4. THE DYNAMIC MODEL INCORPORATING AGE SPECIFIC WAGES AND PRICES

4.1 Theory

The model is now enhanced by allowing wages (and possibly other prices) to vary parametrically with the age of the household. In particular, the labour supply literature suggests that an important element in the work-rate decision is the perceived life cycle wage profile which, for various reasons such as experience, seniority, etc., can be expected to follow a path which is explicitly a non-trivial function of age.

Again with a slight variation in notation, define the optimal value function by:

$$V(w, \pi, t) = \max \int_t^T e^{-\delta(s-t)} g(x(s), p(s))\, ds ,$$

subject to $\dot{w}(s) = r(s)w(s) - x(s)$,

$$w(t) = w ,$$

$$w = \int_t^T e^{-\int_t^s r(v)dv} p(s)'\eta ds + a ,$$

where the functional dependence of p on s is represented by a vector of parameters π.

The advantage of specifying p in this way is that $V(w, \pi, t)$ behaves mathematically in a manner equivalent to the optimal value function of Section 3. However, in this case the interpretation is slightly different in that the explicit dependence of V on t represents two effects - the effect of the ageing process as in the previous section, and the varying nature of wages (or the interest rate and prices) along a profile. The implications of

$$V(w, p, t) = \max \int_t^T e^{-\delta(s-t)} g(x(s), p)\, ds ,$$

subject to $\dot{w}(s) = rw(s) - x(s)$,

$$w(t) = w ,$$

$$w = \int_t^T e^{-r(s-t)} p'\eta ds + a .$$

The Hamilton-Jacobi equation is:

(3.1) $V_w\dot{w} + V_t \equiv \delta V - g$,

which implies:

(3.2a) $V_w = g_x = y$, say,

(3.2b) $V_{ww}\dot{w} + V_w r + V_{wt} = \delta V_w$,

and hence

(3.3a) $\dot{w} = [(\delta-r)V_w - V_{wt}]/V_{ww}$,

(3.3b) $x = rw - [(\delta-r)V_w - V_{wt}]/V_{ww}$.

Defining ψ as V_w^{-1}, the identity $w \equiv \psi(V_w(w, p, t), p, t)$ yields $1 = \psi_y V_{ww}$ and $0 = \psi_y V_{wt} + \psi_t$, so that (3.3a) and (3.3b) become:

(3.4a) $\dot{w} = (\delta-r)y\psi_y + \psi_t$,

(3.4b) $x = r\psi + (r-\delta)y\psi_y - \psi_t$.

Defining ϕ as g_x^{-1}, the relationship $x = \phi(y, p)$ suggests, in view of (3.4b), that ϕ and ψ are related via the differential operator:

(3.5) $\phi = D\,\psi$,

where $D = rI + (r-\delta)y\partial/\partial y - \partial/\partial t$.

For purposes of exposition, the model has been formulated with a zero bequest function in order to concentrate attention on the ageing effect, and this implies the terminal condition $V(w, p, T) = 0$. The main implication of (3.5), over and above (2.8), is that, while $\psi(w, p, t)$ depends explicitly upon t, $\phi(y, p)$ depends upon t only implicitly, through y. This point is best illustrated by means of an example.

3.2 An Example

Consider the generalised dynamic model derived in Section 2.2. Let the function $g(x, p)$ for the finite horizon model of this section be the same as that defined implicitly by (2.23). Thus, let g_x^{-1} be defined by:

(3.6) $$\phi(y, p) = P_o + P_1^{\varepsilon} P_2^{\varepsilon-1} y^{\varepsilon-1} .$$

Since ψ depends explicitly upon t, any conjectured functional form must allow for this. Consider the form:

(3.7) $$\psi(y, p, t) = \theta_o(t)P_o + \theta_1(t)P_1^{\varepsilon} P_2^{\varepsilon-1} y^{\varepsilon-1} .$$

Then applying the differential operator (3.5) yields:

$$D\psi = (r\theta_o - \dot{\theta}_o)P_o + \{[\varepsilon r + (1-\varepsilon)\delta]\theta_1 - \dot{\theta}_1\}P_1^{\varepsilon}P_2^{\varepsilon-1}y^{\varepsilon-1} ,$$

where $\cdot$ denotes d/dt and equality of coefficients between ϕ and $D\psi$ requires:

$$\dot{\theta}_o = r\theta_o - 1 ,$$

$$\dot{\theta}_1 = [\varepsilon r + (1-\varepsilon)\delta]\theta_1 - 1 .$$

These conditions are satisfied by setting:

$$\theta_o(t) = \{1 - e^{r(t-T)}\}/r ,$$

$$\theta_1(t) = \{1 - e^{[\varepsilon r + (1-\varepsilon)\delta](t-T)}\}/[\varepsilon r + (1-\varepsilon)\delta] ,$$

where the constant of integration has been eliminated by the terminal condition $w(T) = 0$. Use of these coefficients allows solution for y as a function of w:

(3.8) $$y = \{\{w - [1 - e^{r(t-T)}]P_o/r\}[\varepsilon r + (1-\varepsilon)\delta]$$
$$\{1 - e^{[\varepsilon r + (1-\varepsilon)\delta](t-T)}\}^{-1}P_1^{-\varepsilon}\}^{\frac{1}{\varepsilon-1}}/P_2 .$$

Use of (3.8) in conjunction with the Frisch demands (2.22) gives a system of generalized intertemporal demand equations parameterized in terms of age of household (through t) and allowing for wealth effects (through w) and for very general price effects (through P_o, P_1, P_2 and w).

A solution procedure more in keeping with the current ORANI implementation would result if (3.6) were to be solved for y as a function of x, and substituted in (2.22) to give the Marshallian demand functions in terms of full expenditure ('consumption') x. The full expenditure function is derived by substituting (3.8) into (3.6). This procedure provides an alternative means by which ORANI may be closed, with the consumption function providing the closure allowing for demographic effects and specified in a way consistent with the specification of the allocation system.

For this example, the consumption function becomes:

(3.9) $$x = P_o + \frac{\varepsilon r + (1-\varepsilon)\delta}{1 - e^{[\varepsilon r + (1-\varepsilon)\delta](t-T)}} \{w - P_o/(\frac{r}{1 - e^{r(t-T)}})\} .$$