Impact Project

Impact Centre
The University of Melbourne
153 Barry Street, Carlton
Vic. 3053 Australia
Phones: (03) 344 7417
Telex: AA 35185 UNIMEL
Telegrams: UNIMELB, Parkville

IMPACT is an economic and demographic research project conducted by Commonwealth Government agencies in association with the Faculty of Economics and Commerce at The University of Melbourne, the Faculty of Economics and Commerce and the Departments of Economics in the Research Schools at the Australian National University and the School of Economics at La Trobe University.

FROM ELES TO ELESA : A LINEAR EXPENDITURE SYSTEM WITH ASSETS

by

Philip D. Adams

University of Melbourne

Preliminary Working Paper No. IP-28 Melbourne November 1986

ISSN 0813 - 7986

ISBN 0 642 10104 3

Merton, R.C. (1971), 'Optimal Consumption and Portfolio Rules in a Continuous-Time Model', Journal of Economic Theory, 3(Dec), pp. 373-413.

Merton, R.C. (1973), 'Erratum', Journal of Economic Theory, 5(April), pp. 213 and 214.

Merton, R.C. (1982), 'On the Mathematics and Economics Assumptions of Continuous-Time Models', in Sharpe, W.F. and C.M. Cootner (eds), Financial Economics, Essays in the Honour of Paul Cootner, Prentice-Hall, Englewood Cliffs, N.J, pp. 19-51.

Powell, A.A. (1974), Empirical Analytics of Demand Systems, Lexington Books, D.C. Heath and Company, Lexington, Mass.

Ramsey, F.P. (1928), 'A Mathematical Theory of Saving', Economic Journal, 38(152), pp. 543-59.

Upcher, M.R. and K.R. McLaren (1986), 'Specification, Identification and Estimation of a Continuous-Time Portfolio Model', mimeo, Department of Econometrics and Operations Research, Monash University.

Tintner, G. and J.K. Sengupta (1972), Stochastic Economics, Academic Press, New York.

CONTENTS

ABSTRACT

The aim of this paper is to introduce financial portfolio behaviour into Lluch's Extended Linear Expenditure System (ELES). Using Merton's model of individual behaviour under uncertainty, a new system is derived which contains a modified form of the ELES and, as a subsystem, a set of portfolio demand equations. The latter are quite simple expressions and imply that the demand for any risky asset is a positive function of the own rate of return relative to the 'safe' rate of return, and diminishes with increases in relative and absolute riskiness.

REFERENCES

Beckmann, M.J. (1968), Dynamic Programming of Economic Decisions, Springer-Verlag, Berlin.

Bellman, R.E. (1957), Dynamic Programming, Princeton University Press.

Clements, K.W. (1976), 'A Linear Allocation of Spending-Power System: A Consumer Demand and Portfolio Model', Economic Record, 52(138), pp. 182-98.

Cox, D.A. and H.D. Miller (1965), The Theory of Stochastic Processes, John Wiley and Sons, New York.

Dreyfus, S.E. (1965), Dynamic Programming and the Calculus of Variations, Academic Press, New York.

Fisher, I. (1930), The Theory of Interest, Yale University Press.

Flemming, J.S. (1974), 'Portfolio Choice and Liquidity Preference: A Continuous-Time Treatment', in Johnson, H.G. and A.R. Nobay (eds), Issues in Monetary Economics, Oxford University Press, pp. 137-50.

Hubbard, R.G. (1983), The Financial Impacts of Social Security: A Study of Effects on Household Wealth Accumulation and Allocation, Monograph Series in Finance and Economics, Salomon Brothers Center for the Study of Financial Institutions, New York University.

Hubbard, R.G. (1985), 'Personal Taxation, Pension Wealth, and Portfolio Composition', Review of Economics and Statistics, 67(1), pp. 53-60.

Ito, K. (1951), 'On Stochastic Differential Equations', Memoirs of the American Mathematics Society, 4, pp. 1-51.

Lluch, C. (1973), 'The Extended Linear Expenditure System', European Economic Review, 4(1), pp. 21-32.

Lluch, C., A.A. Powell and R.A. Williams (1977), Patterns in Household Demand and Saving, Oxford University Press.

Merton, R.C. (1969), 'Lifetime Portfolio Selection Under Uncertainty: The Continuous-Time Case', Review of Economics and Statistics, 51(2), pp. 247-57.

10. The class of utility function considered by Merton contained all members of the HARA (hyperbolic absolute risk-aversion) family, of which the Klein-Rubin form used here is a special case.

11. If Ω is diagonal (i.e., $\sigma(i,j) = 0$ for all $i \neq j$), then (10) becomes more easily interpreted as:

$$w_k^{t*} W^t = \frac{(a_k - a_K)}{\sigma(k,k)\, a_K} \left(\xi^t - \sum_{m=1}^{M} q_m^t \bar{x}_m \right) \quad (k = 1, \ldots, K-1).$$

12. Those elasticities pertaining specifically to the ELES component (such as the commodity price elasticites) are not tabulated as they are well documented elsewhere -- see, for example Powell (1974, ch.6) and Lluch, Powell and Williams (1977, ch.2).

FROM ELES TO ELESA: A LINEAR EXPENDITURE SYSTEM WITH ASSETS*

by

Philip D. Adams

1. INTRODUCTION

In most models of consumer behaviour, total consumption expenditure is treated as a predetermined variable. The maintained position in these models is that the decision of how much to consume now rather than later (the intertemporal problem) is separate from the decision of how to allocate total expenditure among the consumption opportunities of the present (the atemporal problem). Ramsey (1928) and Fisher (1930) were the first to demonstrate, theoretically, that both aspects of the consumer's decision-making process could be handled within the one framework. However, it was not until much later that models emerged which were suitable for econometric implementation. The first of these to be widely used was Lluch's (1973) 'Extended Linear Expenditure System' (ELES), which is the point of reference for this paper.

Lluch takes the utility function associated with Stone's Linear Expenditure System and embeds this into an '... intertemporal utility maximization problem, subject to an expected wealth constraint' (Lluch, 1973, p. 21). The solution to this problem is the ELES and contains equations for the optimal time paths of expenditure on different commodities and for aggregate saving. Only one financial asset, 'non-human wealth', is recognized, and its price and 'rate of reproduction' are assumed to be constant over the consumer's plan.

In this paper we show how Lluch's model can be generalised to accommodate a consumer who must not only decide on how much to save in aggregate, but also how to allocate total savings amongst a menu of assets in a world where asset prices are not constant, but are subject to unpredictable variations.[1] Thus the consumer's portfolio and consumption behaviour are considered simultaneously, and optimal rules for financial asset holdings are derived as well as for consumption expenditures. The resulting system is called the Extended Linear Expenditure System with Assets (hereinafter ELESA).

2. THE MODEL

The approach taken in this paper to model the decision-making process of the consumer is based on the model of individual behaviour under uncertainty developed in Merton (1969, 1971 and 1973); to which the reader should refer for further details.[2] Risk plays an important role in this framework, and is introduced by assuming that returns on financial assets are stochastic processes. Following Merton, our model is specified in continuous-time and uses a utility function which recognizes explicitly that utility is generated by consumption, not by wealth. Thus we remain true to the postulate of 'intrinsic worthlesness' -- no financial asset is wanted in and of itself, but only for the future consumption it supports.

Consider an individual, in a world of perfect financial markets, whose portfolio consists of K assets and whose income is generated by capital gains and the yield on an exogenously given endowment of human wealth.[3] Let instant h be an arbitrary point of time in the consumer's planning horizon extending out from the current instant t. The expectation held for the price of financial asset k at h, p_k^h, is assumed to be generated by the following stationary (over planning time) log-normal process:

$$\frac{dp_k^h}{p_k^h} = a_k\,dh + \sigma_k\,dz_k^h \qquad (k = 1,\dots, K) \quad (1),$$

where a_k is the expected instantaneous 'permanent' proportional change in the price of financial asset k, per unit time; and σ_k^2 is the instantaneous variance, per unit time. The process dz_k^h is a normally distributed random variable with mean zero and variance dh, and is the continuous-time analogue of the difference equation defining a simple random walk.[4]

A **permanent** change in price means long-run, or average, change per unit time. The assumption being made is that the expected change in price of an asset, at each point of the consumer's plan, consists of two

6. The assumption of a risk-free asset makes the problem at hand much more tractable (see Merton, 1971, pp. 379-88). In Upcher and McLaren (1986) this assumption is dropped since the existence of inflation is explicitly accounted for. Their analysis, as a consequence, is restricted to the constant relative risk aversion class of utility function. Note also, that in (3), the w_k^h ($k = 1,\dots, K-1$) are unconstrained because the relation $w_K^h = \sum_{k=1}^{K-1} w_k^h$ will ensure that the constraint on the sum of the w_k^h's is at all times satisfied.

7. A heuristic description of dynamic programming is given in Dreyfus (1965). The merits (or otherwise) of using this technique as a tool for solving certain economic problems, are discussed in Beckmann (1968), and Tintner and Sengupta (1972).

8. Note that **Ω**, where

$$\mathbf{\Omega} = \begin{bmatrix} \sigma_{(1,1)} & \cdots & \sigma_{(1,K-1)} \\ \vdots & \ddots & \vdots \\ \sigma_{(K-1,1)} & \cdots & \sigma_{(K-1,K-1)} \end{bmatrix},$$

is assumed to be non-singular ($\sigma_{(i,j)}$ is defined in the Appendix). According to Merton (1971, p. 382; and 1973), a sufficient condition for a unique interior maximum is that $S_{WW}(\dots, \tau) < 0$ (i.e., that $S(\dots, \tau)$ is strictly concave in W). Satisfaction of this condition is assumed for the remainder of this paper.

9. The implications of these three particularizations are discussed in Powell (1974, ch.6).

ENDNOTES

* I wish to thank Peter Dixon, Keith McLaren (especially for bringing to my attention, and helping to interpret, the work of Merton) and Alan Powell for many helpful comments. All remaining errors are my own.

1. Clements (1976) also incorporates consumer portfolio behaviour into the ELES framework. However, the methodology used by Clements, unlike that used in this paper, is purely deterministic and involves the transformation of an (unspecified) intertemporal choice problem into a (specified) single-period problem using a 'collapsibility' theorem.

2. The relative merits of the Merton model are considered in Flemming (1974); while recent examples of applications include Hubbard (1983 and 1985), and Upcher and McLaren (1986).

3. In the ELES, there is no possibility of capital gains from holding 'non-human wealth', since the price of that asset is identically equal to one.

4. Random walks, in both discrete and continuous time, are described in Cox and Miller (1965). The distributions defined in (1) are also known as Ito processes -- see Ito (1951). The reader should note that equation (1) is not differentiable in the usual sense; instead a more general differential calculus of the type described in Merton (1982) must be used.

5. For a rigorous derivation of (2) from its discrete-time analogue, see Merton (1971, pp. 377-9).

independent parts: one permanent and the other transitory. The transitory part is captured by the random walk $\sigma_k dz_k$.

Given the stochastic nature of (1), the consumer's accumulation relationship for non-human wealth (hereinafter 'wealth', the planned value of which for h is denoted W^h) must be expressed as the stochastic differential equation:

$$dW^h = \sum_{k=1}^{K} dp_k^h D_k^h + (y^h - x^{h\prime} q^h)dh \qquad (2),$$

where D_k^h is the number of units of financial asset k expected (from the viewpoint of the consumer at t) to be held at the notional instant h (a liability is represented by a negative value for D_k^h); y^h is the exogenously given instantaneous flow of expected labour income at h, per unit time; x^h is an M length vector containing the rates at which each item of the budget is expected to be consumed at instant h, per unit time; and q^h is an M vector of expected commodity prices (note that a prime (') is used to denote the transpose of a column vector).[5]

Let one of the financial assets, say asset K, be risk-free (i.e. $\sigma_K = 0$). Then, upon substitution of (1), equation (2) becomes:

$$dW^h = \sum_{k=1}^{K-1} [(a_k - a_K)w_k^h W^h \, dh + \sigma_k w_k^h W^h \, dz_k^h] + (a_K W^h + y^h - x^{h\prime} q^h) \, dh \qquad (3),$$

where w_k^h is the proportion of wealth W^h expected to be invested in asset k at h ($\sum_{k=1}^{K} w_k^h = 1$).[6]

At t, the consumer's intertemporal choice problem can be expressed as follows. Let $U(x^h, h)$ be the instantaneous intertemporal utility function which is supposed to include only planning time and the time rate of demand for each of the M consumables as arguments. The consumer, who is assumed to have an infinite planning horizon, seeks to maximize the criterion functional:

$$E_t \int_t^{\infty} U(x^h, h) \, dh \quad ,$$

subject to:

- exogenously given expected time paths for labour earnings $\{y^h\}$

and commodity prices $\{\mathbf{q}^h\}$, both of which are assumed to be stationary over the plan and set equal to their actual values at t $[y^h = y^t,\ \mathbf{q}^h = \mathbf{q}^t;\ h \in [t,\infty)]$;

- an initial endowment of non-human wealth W^t;

and

- the accumulation relationships, equations (1) and (3).

The utility function U( ) is assumed to be strictly concave in $\mathbf{x}^h$. E_t is the expectation operator, conditional upon knowing all relevant information at, or before, t.

To derive optimal rules for each x_m ($x_m \in \mathbf{x}$; m = 1,..., M) and for w_k (k = 1,..., K) (i.e., functions associating values for every x_m and w_k with $p_1, p_2, \ldots, p_K$ and W, at every point h of planning time), the technique of stochastic dynamic programming can be used.[7] This technique allows the following theorem to be stated (paraphrased from Merton, 1971, p. 381).

> If p_k for all k are generated by strong diffusion processes (i.e., Markov processes in which only continuous changes of state occur, such as the Ito processes of (1)), U is strictly concave in $\mathbf{x}$, then there exists a set of optimal control rules $\{\mathbf{x}^{\tau*}, w_1^{\tau*}, \ldots, w_{K-1}^{\tau*}\}$, and these rules satisfy
>
> $$0 = \theta(\mathbf{x}^{\tau*}, w_1^{\tau*}, \ldots, w_{K-1}^{\tau*}; W^\tau, \tau) \geq \theta(\mathbf{x}^{\tau}, w_1^{\tau}, \ldots, w_{K-1}^{\tau}; W^\tau, \tau)$$
>
> for $\tau \in [t,\infty)$, where θ( )is the fundamental differential equation derived in the Appendix.

To find values for $\mathbf{x}^\tau$ and w_k^τ (k = 1,..., K-1) which maximize the fundamental differential equation, we set the derivatives of θ( ) to zero, thereby obtaining the following necessary conditions:

Because financial asset prices are stationary over the plan (i.e., a and σ are independent of τ for all k), S(.., τ) will be a function of W^τ and τ, but not P_k^τ (k=1,..., K). Thus equation (A5) reduces to:

$$0 = \max_{\{\mathbf{x}^\tau, w_1^\tau, .., w_{K-1}^\tau\}} \Big[U(\mathbf{x}^\tau, \tau) + S_W(\ldots, \tau)[\sum_{k=1}^{K-1} (a_k - a_K) w_k^\tau W^\tau + (a_K W^\tau + y^t - \mathbf{x}^{\tau\prime} \mathbf{q}^t] + S_\tau(\ldots, \tau) + \frac{1}{2} S_{WW}(\ldots, \tau)[\sum_{i=1}^{K-1}\sum_{j=1}^{K-1} w_i^\tau w_j^\tau \sigma_{(i,j)} W^{\tau\,2}] \Big]$$

$$= \max_{\{\mathbf{x}^\tau, w_1^\tau, .., w_{K-1}^\tau\}} \Big[\theta(\mathbf{x}^\tau, w_1^\tau, .., w_K^\tau; W^\tau, \tau) \Big] \qquad \text{(A6)},$$

which is the **fundamental differential equation** for the problem being considered here.

$$0 = \max_{\{\mathbf{x}^s, w_1^s, .., w_{K-1}^s\}} E_\tau \Big[U(\mathbf{x}^\tau, \tau)\,\Delta h + S_W(\,..,\tau)[W^{\tau+\Delta h} - W^\tau] + S_\tau(\,..,\tau)\,\Delta h$$

$$+ \sum_{k=1}^{K-1} S_{p_k}(\,..,\tau)\,[p_k^{\tau+\Delta h} - p_k^\tau] + \frac{1}{2} S_{WW}(\,..,\tau)[\,W^{\tau+\Delta h} - W^\tau]^2$$

$$+ \frac{1}{2} \sum_{i=1}^{K-1}\sum_{j=1}^{K-1} S_{p_i p_j}(\,..,\tau)\,[\,p_i^{\tau+\Delta h} - p_i^\tau\,][\,p_j^{\tau+\Delta h} - p_j^\tau\,]$$

$$+ S_{W\tau}(\,..,\tau)[W^{\tau+\Delta h} - W^\tau]\,\Delta h + \sum_{k=1}^{K-1} S_{p_k\tau}(\,..,\tau)[p_k^{\tau+\Delta h} - p_k^\tau]\Delta h$$

$$+ \sum_{k=1}^{K-1} S_{p_k W}(\,..,\tau)\,[p^{\tau+\Delta h} - p^\tau][W^{\tau+\Delta h} - W^\tau] + o(\Delta h) \Big]$$

$$(s \in [\tau, \tau+\Delta h]) \quad (A4),$$

where the notation used for the partial differentials is: $S_W(\,..,\tau) = \partial S(W^\tau, p_1^\tau, ..., p_K^\tau, \tau)\,/\partial W^\tau$, $S_{p_k}(\,..,\tau) = \partial S(W^\tau, p_1^\tau, ..., p_K^\tau, \tau)\,/\partial p_k$, $S_\tau(\,..,\tau) = \partial S(W^\tau, p_1^\tau, ..., p_K^\tau, \tau)\,/\partial \tau$, $S_{WW}(\,..,\tau) = \partial^2 S(W^\tau, p_1^\tau, ..., p_K^\tau, \tau)\,/\partial W^\tau \partial W^\tau$, etc.

Taking expectations, and then dividing by Δh and letting Δh approach zero, produces:

$$0 = \max_{\{\mathbf{x}^\tau, w_1^\tau, .., w_{K-1}^\tau\}} \Big[U(\mathbf{x}^\tau, \tau) + S_W(\,..,\tau)[\sum_{k=1}^{K-1}(a_k - a_K) w_k^\tau W^\tau + (a_K W^\tau + y^t - \mathbf{x}^{\tau\prime} \mathbf{q}^t)]$$

$$+ \sum_{k=1}^{K-1} S_{p_k}(\,..,\tau)\, p_k^\tau a_k + S_\tau(\,..,\tau)$$

$$+ \frac{1}{2} S_{WW}(\,..,\tau)[\sum_{i=1}^{K-1}\sum_{j=1}^{K-1} w_i^\tau w_j^\tau \sigma_{(i,j)} (W^\tau)^2\,]$$

$$+ \frac{1}{2} \sum_{i=1}^{K-1}\sum_{j=1}^{K-1} S_{p_i p_j}(\,..,\tau)\,[\,p_i^\tau p_j^\tau \sigma_{(i,j)}\,]$$

$$+ \sum_{i=1}^{K-1}\sum_{j=1}^{K-1} S_{p_i W}(\,..,\tau)[\,w_j^\tau W^\tau p_i^\tau \sigma_{(i,j)}\,] \Big] \quad (A5).$$

$$x_m^{\tau *} q_m^t = G_m(S_W(\,..,\tau), \tau) \qquad (m = 1, ..., M) \quad (4),$$

and

$$w_K^{\tau *} = - \frac{S_W(\,..,\tau)}{S_{WW}(\,..,\tau) W^\tau} \sum_{i=1}^{K-1} \sigma^{(i,k)}(a_i - a_K) \qquad (k = 1, ..., K-1) \quad (5),$$

in which: $G_m(\ )$ is the inverse function $\{x_m = [\partial U(\ )/\partial x_m]^{-1}\}$; $\sigma^{(i,j)}$ is the (i,j)'th element of $\mathbf{\Omega}^{-1}$(where $\mathbf{\Omega}$ is the symmetric (K-1 x K-1) covariance matrix of financial asset returns); and the remainder of the notation including the interpretation of the optimal expected value function $S(\,..,\tau)$, is explained in the Appendix.[8]

To solve for $\mathbf{x}^{\tau *}$ and $w_k^{\tau *}$ $(k = 1, ..., K-1)$, (4) and (5) are substituted into $\theta(\ )$ which becomes a second-order partial differential equation in $S(\,..,\tau)$. Having then solved this equation for $S(\,..,\tau)$ we substitute back into (4) and (5) to obtain the required optimal rules.

Substituting for (4) and (5) into $\theta(\ )$, gives:

$$0 = \Big[U(\frac{G_1}{q_1^\tau}, ..., \frac{G_M}{q_M^\tau}, \tau) + S_\tau(\,..,\tau)$$

$$+ S_W(\,..,\tau)[\,a_K W^\tau + y^t - \sum_{m=1}^{M} G_m\,]$$

$$- \frac{S_W(\,..,\tau)^2}{2\, S_{WW}(\,..,\tau)} \sum_{i=1}^{K-1}\sum_{j=1}^{K-1} [\sigma^{(i,j)}(a_i - a_K)(a_j - a_K) \Big] \quad (6).$$

Before (6) can be solved for $S(\,..,\tau)$, $U(\ )$ must be given an explicit form. Already having adopted intertemporal additivity of the utility functional, as in Lluch (1973) we adopt the following additional simplifying particularizations:

- stationarity of the instantaneous utility function;
- constancy of the time-preference discount rate, δ ;

and

- the Klein-Rubin (or Stone-Geary) form for the instantaneous utility function.[9]

Thus

$$U(\mathbf{x}^\tau, \tau) = \exp(-\delta(\tau - t))\, V(\mathbf{x}^\tau) ,$$

in which

$$V(\mathbf{x}^\tau) = \sum_{m=1}^{M} \beta_m \log(x_m^\tau - \bar{x}_m) ,$$

where the β's are positive paramaters summing to unity, and $\bar{x}_m$ is, as conventionally interpreted, the time-invariant instantaneous flow of consumption of commodity m necessary for subsistence.[10] Substitution into (6) yields:

$$0 = \Bigg[\exp(-\delta(\tau - t))[\sum_{m=1}^{M} \beta_m \log(\beta_m / S_W(\ldots, \tau) q_m^t) - \delta(\tau - t) - 1] + S_\tau(\ldots, \tau) + S_W(\ldots, \tau)[a_K W^\tau + y^t - \sum_{m=1}^{M} q_m^t \bar{x}_m] - \frac{S_W(\ldots, \tau)^2}{2\, S_{WW}(\ldots, \tau)} \sum_{i=1}^{K-1}\sum_{j=1}^{K-1} [\sigma^{(i,j)}(a_i - a_K)(a_j - a_K) \Bigg] \quad (7).$$

A solution to (7) is obtained by choosing

$$S(\ldots, \tau) = \frac{\exp(-\delta(\tau - t))}{\delta} \log(\xi^\tau - \sum_{m=1}^{M} q_m^t \bar{x}_m) + h(\tau) \quad (8),$$

in which $\xi^\tau = a_K W^\tau + y^t$, is the expected instantaneous flow of 'safe' income at τ (i.e., the expected instantaneous flow of income from human and non-human sources, where the latter is imputed at the 'safe' rate of return a_K), and $h(\tau)$, undefined, is a function of τ.

With our attention focussed on the initial instant of planning time $\tau = t$, the optimal consumption and portfolio rules can now be written in their final form, as:

$$\text{Variance}(W^{\tau+\Delta h} - W^\tau) = \sum_{i=1}^{K-1}\sum_{j=1}^{K-1} w_i^\tau w_j^\tau \sigma_{(i,j)} (W^\tau)^2 \Delta h ,$$

$$\text{Covariance}(p_j^{\tau+\Delta h} - p_j^\tau)(W^{\tau+\Delta h} - W^\tau) = \sum_{i=1}^{K-1} w_i^\tau W^\tau p_j^\tau \sigma_{(i,j)} \Delta h \quad (j = 1, \ldots, K-1),$$

in which $E_\tau(dz_i^\tau\, dz_j^\tau) = \rho_{(i,j)} \Delta h$, and $\sigma_{(i,j)} = \rho_{(i,j)} \sigma_i \sigma_j$, where $\rho_{(i,j)}$ is the correlation coefficient between the processes dz_i^τ and dz_j^τ. It is assumed that all higher moments are $\mathbf{o}(\Delta h)$ (i.e., are moments which, after division by Δh, approach zero as Δh approaches zero).

Bellman's 'principle of optimality' for optimal feedback problems (Bellman, 1957) implies the following recursive relation:

$$S(W^\tau, p_1^\tau, \ldots, p_K^\tau, \tau) = \max_{\{\mathbf{x}^s, w_1^s, \ldots, w_{K-1}^s\}} E_\tau \Big[U(\mathbf{x}^\tau, \tau)\, \Delta h + \mathbf{o}(\Delta h) + S(W^{\tau+\Delta h}, p_1^{\tau+\Delta h}, \ldots, p_K^{\tau+\Delta h}, \tau+\Delta h) \Big] \quad (s \in [\tau, \tau+\Delta h]) \quad (A3),$$

in which $W^{\tau+\Delta h}$, $p_1^{\tau+\Delta h}, \ldots, p_K^{\tau+\Delta h}$ are given by equations (A1) and (A2); $U(\mathbf{x}^\tau, \tau)\,\Delta h + \mathbf{o}(\Delta h)$ is an approximation of $\int_\tau^{\tau+\Delta h} U(\mathbf{x}^h, h)\, dh$; and E_τ is the conditional expectation operator, conditional upon knowing W^τ, $p_1^\tau, \ldots, p_K^\tau$. The set $\{\mathbf{x}^s, w_1^s, \ldots, w_{K-1}^s\}$ contains all admissible control rules emanating from τ (given W^τ, $p_1^\tau, \ldots, p_K^\tau$) and extending out to $\tau+\Delta h$.

Taylor series expansion of the right hand side of (A3) about $(W^\tau, p_1^\tau, \ldots, p_K^\tau, \tau)$ and cancellation of $S(W^\tau, p_1^\tau, \ldots, p_K^\tau, \tau)$ yields:

APPENDIX

A DIGRESSION ON STOCHASTIC PROGRAMMING: DERIVATION OF THE FUNDAMENTAL DIFFERENTIAL EQUATION

The following derivation draws upon Dreyfus (1965). For the problem being considered here we define a function S( ), called the **optimal expected value function**, such that:

$$S(W^{\tau}, p_1^{\tau},\dots, p_K^{\tau}, \tau) \overset{def}{=} \underset{\{\mathbf{x}, w_1, \dots, w_{K-1}\}}{\text{Max}} E_{\tau} \int_{\tau}^{\infty} U(\mathbf{x}^h, h)\, dh \quad ,$$

where the set $\{\mathbf{x}, w_1, \dots, w_{K-1}\}$ contains all admissible control rules emanating from an arbitrary instant of (planning) time τ ($\tau \geq t$), given W^{τ}, $p_1^{\tau},\dots, p_K^{\tau}$.

To derive $S(W^{\tau}, p_1^{\tau},\dots, p_K^{\tau}, \tau)$, we must first consider the problem in discrete time, with time increments of Δh. We start by rewriting equations (1) and (3) as:

$$p_k^{\tau+\Delta h} - p_k^{\tau} = p_k^{\tau} \left[a_k \Delta h + \sigma_k (z_k^{\tau+\Delta h} - z_k^{\tau}) \right] \qquad (k = 1,\dots, K) \quad \text{(A1)},$$

and

$$W^{\tau+\Delta h} - W^{\tau} = \sum_{k=1}^{K-1} (a_k - a_K) w_k^{\tau} W^{\tau} \Delta h + (a_K W^{\tau} + y^t - \mathbf{x}^{\tau\prime} \mathbf{q}^t) \Delta h + \sum_{k=1}^{K-1} \sigma_k w_k^{\tau} W^{\tau} (z_k^{\tau+\Delta h} - z_k^{\tau}) \qquad \text{(A2)}.$$

Notice that the first and second moments of $(p_k^{\tau+\Delta h} - p_k^{\tau})$ and $(W^{\tau+\Delta h} - W^{\tau})$ are:

$$E(p_k^{\tau+\Delta h} - p_k^{\tau}) = p_k^{\tau} a_k \Delta h \qquad (k = 1,\dots, K),$$

$$E(W^{\tau+\Delta h} - W^{\tau}) = \sum_{k=1}^{K-1} (a_k - a_K) w_k^{\tau} W^{\tau} \Delta h + (a_K W^{\tau} + y^t - \mathbf{x}^{\tau\prime} \mathbf{q}^t) \Delta h \quad ,$$

$$\text{Covariance}(p_i^{\tau+\Delta h} - p_i^{\tau})(p_j^{\tau+\Delta h} - p_j^{\tau}) = p_i^{\tau} p_j^{\tau} \sigma_{(i,j)} \Delta h \qquad (i,j = 1,\dots, K-1),$$

$$x_m^{t*} q_m^t = \bar{x}_m q_m^t + \beta_m \mu \left(\xi^t - \sum_{m=1}^{M} \bar{x}_m q_m^t \right) \qquad (m = 1,\dots, M) \quad (9),$$

and

$$w_K^{t*} W^t = \frac{\sum_{i=1}^{K-1} \sigma^{(i,k)} (a_i - a_K)}{a_K} \left(\xi^t - \sum_{m=1}^{M} \bar{x}_m q_m^t \right) \qquad (k = 1,\dots, K-1) \quad (10),$$

where, by definition, $\mu = \delta / a_K$.

Equations (9) and (10) are the central results of this paper and, together, form the ELESA. Equation (9) is Lluch's ELES, with the single market rate of interest (ρ in Lluch's notation) replaced by the safe rate of return a_K and permanent income (z) equal to ξ. Equation (10) explains the consumer's portfolio behaviour as a system of linear functions in 'safe' income. Consistent with the more traditional mean-variance analysis of portfolio behaviour, equation (10) implies that the demand for risky asset k is an increasing function of the weighted sum of rates of return on each risky asset (a_i ; i = 1,..., K-1) relative to the safe rate of return a_K . The weight $\sigma^{(i,k)}$ gives an indication of the relative security endowed from investing in k rather than i.[11]

To assist the reader in further understanding the ELESA, certain elasticity formulae for the system are provided in Table 1.[12] Keep in mind though, that these formulae are derived from the continuous-time system, and so care is needed when interpreting them with discrete-time data. For example, ξ^t is a function of W^t , which is itself a function of asset prices at t. In discrete-time, changes in these prices may induce, over the unit time interval, a change in W^t and thus ξ^t . This effect is additional to that reported in Table 1 and would need explicit treatment in an operational version of the theory using discrete-time data.

3. CONCLUSIONS

In this paper we have shown how Lluch's ELES can be generalised to accommodate the portfolio behaviour of a consumer whose perception of future asset prices is uncertain. The resulting system, called the ELESA,

Table 1: Formulae for Elasticities in the ELESA

Description	Formula[a]	Comments[b]
Elasticity of demand for commodity m (risky asset k) at t with respect to:		
'Safe' income ξ^t	$E_m = \frac{\beta_m \mu \xi^t}{x_m^t q_m^t}$	Always positive.
	$E_k = \frac{\sum_{i=1}^{K-1} \sigma^{(i,k)}(a_i - a_K)\xi^t}{a_K w_K^t W^t}$	Positive if k not a liability, and the sum of $\sigma^{(i,k)}(a_i - a_K) \geq 0$.
Permanent change in price of asset K a_K^t	$\eta_{(m,K)} = - \frac{\beta_m (y^t - \sum_{m=1}^{M} q_m^t \bar{x}_m)}{a_K x_m^t q_m^t}$	Negative if $y^t \geq$ expenditure on subsistence.
	$\eta_{(k,K)} = - \frac{\sum_{i=1}^{K-1} \sigma^{(i,k)} a_K}{w_K^t}$	Negative if k not a liability, and $\sum_{i=1}^{K-1} \sigma^{(i,k)} \geq 0$.
Permanent change in price of risky asset j a_j^t	$\eta_{(m,j)} = 0$	
	$\eta_{(k,j)} = \frac{\sigma^{(j,k)} a_j (\sup^t)}{a_K w_K^t W^t}$	Positive if $\sup^t \geq 0$, j not a liability, and $\sigma^{(j,k)} \geq 0$.

a $\sup^t = \xi^t - \sum_{m=1}^{M} q_m^t \bar{x}_m$.

b Assuming that for all k, $a_K \geq 0$.

has two basic components. One is a modified form of the ELES, while the other contains equations which determine the consumer's optimal holding of each financial asset.

The ELESA is built upon Merton's model of consumer behaviour under uncertainty (with a specific intertemporal utility function, the Klein-Rubin). As such it inherits the proven flexibility of the latter with respect to alternative asset price dynamics and various (possibly stochastic) streams of wage income (see Merton, 1971). The ELESA, therefore, represents a very general treatment of the consumer decision problem, a treatment which offers considerable scope for empirical applications.