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Evaluating Innovations in Long-Run Closures

THE BEHAVIOUR OF THE MARKET

Impact
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Abstract
REFERENCES

Arye, E. (1959). The Ohio State University's, Ohio State University, Columbus, Ohio.


MacKintosh, G. (1995) The Ohio State University's, Ohio State University, Columbus, Ohio.

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conditions (and in particular, consumer demand where the values of inter-product substitution elasticities are typically less than one) will limit flip-flop. Similarly, for import-competing goods, Armington elasticities typically in the range 0.5–5.0 will tend to attenuate any tendency towards corner solutions. For non-land-using endogenous export industries, however, the potential problem of volatile output responses remains. The relevant commodities in ORANI are minerals or processed minerals (see Bruce (1985)):

- Ferrous Metal Ores (-10.0)
- Non-Ferrous Metal Ores (-8.0)
- Coal (-20.0)
- Basic Non-Ferrous Metals (-10.0)

The number in parenthesis is the export demand elasticity in the parameter file of the 1977–78 (typicalized agriculture) ORANI database. The database does not recognize the existence of a factor specific to the industries producing these commodities which is fixed in the long run.

Having identified the potential problem, it should not be exaggerated. In Table 1 are reproduced results from Horridge (1985) demonstrating that the most volatile of all industries in long-run simulations of a tariff charge are miners. Nevertheless, it should be noted that the orders of magnitude involved are probably not so high as to be implausible a priori to all readers.

The aim of the present paper is to investigate the extent to which the theory of optimal extraction of a mineral may cast light on the above issue. Specifically, are there changes that should be made to the ORANI theory and/or database in order to make the long-run simulated behaviour of the mining industries more plausible? We attempt to answer this question in four stages. First, in Section 2, the relevant general analytical insights are reviewed. Then in Section 3, a special case, hopefully relevant to Australia, is developed in an operational form. Our first efforts to implement this operational form are reported in Section 4. Thence in Section 5 we reach the final stage of our analysis, wherein the implications of the role of mining in long-run closures of ORANI are explored. Finally, in Section 6, brief concluding remarks and a perspective for future research are offered.

\[(A.5) \quad \frac{\partial^2 PV}{\partial x_2 \partial x_3} = \frac{\partial}{\partial x_2} \left( \frac{1}{x_2} \cdot \frac{\partial y'}{\partial x_2} \right) < 0.\]

Now from (3.6),

\[(A.6) \quad \frac{\partial}{\partial x_2} = - \gamma y e^{-r x_2},\]

while from (2.2), (3.2), (3.1) and (3.3),

\[(A.7) \quad \frac{\partial}{\partial x_2} = \frac{3}{2} e^{-r x_2} \left( n x_2 y'(x) \right) - a - by'(x) - \frac{1}{2} \theta [y'(x)]^2 - \gamma y y(x)\]

\[-y y'(x)\]

\[= - \gamma y y'(x)\]

\[= - \gamma y y'(x)\]

\[= - \gamma y y'(x)\]

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THE LONG RUN IS DEFINED BY THE ECONOMICS OF MODERNIZATION, OR HIGHER ORDER OF SITUATION. SEE HORTHOLO (1965), ECONOMICS P. 13.

Order of 20 years. See Horthoft (1965), Economics, p. 13. A possible long-term interpretation of the economy in terms of productive forces and wealth. In the long run, productive forces are defined by the above equations. The long run here is defined by the long-run cost of modernization.

1.0
3.0
4.5
6.0
7.5
9.0
11.0
12.5
15.0
17.5
20.0
25.0
30.0
35.0
40.0
45.0
50.0
55.0
60.0
65.0
70.0
75.0
80.0
85.0
90.0
95.0
100.0

DIFFERENTIATION MEASURE, WE OBTAIN:

\[
\frac{d^2}{dx^2} \left( \int_0^x f(t) \, dt \right) = \frac{d}{dx} \left( \frac{d}{dx} \left( \int_0^x f(t) \, dt \right) \right)
\]

A FISHER-PRICE CONDITION FOR \( \chi \):

1. In the context of the assumptions made, section 3, solve for \( \chi \) the utility function to be maximized to maximize the present value \( (\Pi) \) of the project. Given \( c = \frac{C}{\Pi} + \frac{1}{\Pi} \), then \( \chi \) is a function of known parameters. First, use the initial condition \( \chi(0) = 0 \) in (3.170) to obtain \( \chi(0) \) from Table 1.

TABLE 1

<table>
<thead>
<tr>
<th>Industry</th>
<th>Productivity</th>
<th>Market Price</th>
<th>Output</th>
<th>Profit Margin</th>
</tr>
</thead>
<tbody>
<tr>
<td>Textile</td>
<td>High</td>
<td>Low</td>
<td>Low</td>
<td>High</td>
</tr>
<tr>
<td>Automotive</td>
<td>Medium</td>
<td>Medium</td>
<td>High</td>
<td>Low</td>
</tr>
<tr>
<td>Chemical</td>
<td>Low</td>
<td>High</td>
<td>High</td>
<td>Low</td>
</tr>
</tbody>
</table>

SUMMARIZED BY A ZERON CAN'T CREATE THE MODEL RIGHT OUT. INDUSTRIAL EFFICIENCY IN LONG-RUN (1968) TABLE 1

APPENDIX

PLAN IN THE CASE OF THE QUADRATURE ONE COULD

SUCCESSFUL FROM THE THIRD-ORDER CONDITION TO LOCATE A MAXIMUM FORM.

"
2. REVIEW OF THE CALCULUS-OF-VARIATIONS TREATMENT OF EXHAUSTIBLE RESOURCES

2.1 Basic Ideas

This review aims to give a detailed explanation of the principal results reported in Leivard and Liviatan (1977). Following Hildebrand (1965), the classical calculus of variations problem may be formulated as find a continuously differentiable function \( y(x) \) which maximizes the functional

\[
I = \int_{x_1}^{x_2} F(x, y, y') \, dx,
\]

subject to some end-point constraints (yet to be defined) involving \( x_1 \) and \( x_2 \). To fix ideas in the mining context, let \( x \) be (planning) time, \( y \) be cumulative extractions from a given deposit, and \( y' = \frac{dy}{dx} \) be the rate of extraction. Then if \( x_1 = 0 \) and

\[
F = e^{-r(x-x_1)} R(x, y, y'),
\]

where \( r \) is the discount rate, and \( R \) is the (instantaneous) rate of flow of profits, we can identify \( I \) as the present value of the profit stream.

The end-point constraints binding (2.1) may take a variety of forms, including:

\[
y(x_1) = y_1, \quad y(x_2) = y_2; \text{ with the values of } x_1, x_2, y_1, y_2 \text{ known};
\]

and

\[
y(x_1) = y_1, \quad y(x_2) = g(x_2); \text{ with the values of } x_1, y_1, \text{ and the form and parameters of } g \text{ known}.
\]

In his classic article on the subject, Hotelling (1931) assumed (a) that the rate of output \( y' \) at the terminal point \( (y_2, x_2) \) is zero; and (b) that the deposit is exhausted at the terminal date; i.e., that \( y_2 = y(x_2) = a \), where \( a \) is the initial mineral endowment of the given deposit.

right of (5.3) consistent with a value of \( \eta(10) \) generated by the procedures discussed above in Sub-sections 5.1 and 5.2. Changes in the fortunes of the particular mining industry involved then show up as changes in the rental to the new fixed factor. (In a more ambitious framework, investment in exploration in the projected year would respond to such changes in long-run profitability.)

6. CONCLUDING REMARKS

Long-run supply elasticities in ORANI, as presently configured, may be implausibly high in the case of some mineral products. This paper has developed a model for the optimal exploitation of a mineral resource under conditions where:

(a) known reserves are plentiful; but
(b) mines operate under conditions of cumulative cost increases as mining proceeds.

At this stage, it is not yet known whether the meagre supply of information available will be enough to enable this model to put useful limits on contemporaneous comparative static supply elasticities at an arbitrary future date (say 10 years). A procedure for investigating this issue has been suggested above in Sub-sections 3.1 and 5.2. If the suggested procedure turns out to be feasible, then the database of long-run ORANI can easily be modified to make ORANI behave 'as if' the mining firms behave optimally along the lines described in Sections 2 and 3 of the paper. This amounts to introducing a new factor into the model which, like the different categories of agricultural land, is fixed, even in the long run -- this factor might be loosely thought of as "the known supply of reserves". By ascribing an appropriate share of value added to this factor in the model's database for the year projected, it is possible to ensure that the long-run comparative static behaviour of the mining industries is consistent with the micro theory developed in this paper.

Clearly, the proposals made in this paper are feasible, but a large volume of additional work is required.
\[
0 = x \left( \frac{d}{d\theta} \frac{d}{d\theta} \frac{d}{d\theta} \right) 0 = 0 \quad (9.2)
\]

where \( \theta \) is the time-independent parameter of the function. Let \( \theta = \theta(x) \) then

\[
x \left( \frac{d}{d\theta} \frac{d}{d\theta} \frac{d}{d\theta} 0 \right) = 0 \quad (9.2)
\]

\[
x \left( \frac{d}{d\theta} \frac{d}{d\theta} \frac{d}{d\theta} 0 \right) = f \quad (9.2)
\]

The Lagrangean expression for the Hamiltonian (1969, p. 41) is

\[
\frac{d}{d\theta} \frac{d}{d\theta} \frac{d}{d\theta} 0 + x \frac{d}{d\theta} \frac{d}{d\theta} \frac{d}{d\theta} 0 = f \quad (9.2)
\]

In order to find the Lagrangean expression, we replace \( \phi \) by \( \phi(x) \) in equation (9.2) to obtain the Lagrangean expression for the Hamiltonian function.

The Lagrangean expression for the Hamiltonian function is

\[
x = x \left( \frac{d}{d\theta} \frac{d}{d\theta} \frac{d}{d\theta} \right) 0 \quad (9.2)
\]

The Lagrangean expression for the Hamiltonian function is

\[
x = x \left( \frac{d}{d\theta} \frac{d}{d\theta} \frac{d}{d\theta} \right) 0 \quad (9.2)
\]
(2.9) \[ y(x_x) \leq a, \]

where \( \delta y \) is the variation in the function \( y \), and \( (\delta y') \) is the induced variation in the derivative of that function; viz;

\[ (2.10) \quad (\delta y') = \frac{d}{dx} (\delta y), \]

since the variation of a derivative is equal to the derivative of the variation (Hildebrand (1965, p. 133)). In (2.8), \( H \) is viewed as a function with explicit arguments \( x, y, y', \) and \( \lambda \); i.e., \( H = H (x, y, y', \lambda) \) (see (2.7')). The partials in (2.8) are all evaluated at fixed values of \( x \) and \( \lambda \); \( \frac{\partial H}{\partial y} \) is evaluated with \( y' \) held fixed and \( \frac{\partial H}{\partial y'} \) with \( y \) held fixed. That is, \( \frac{\partial H}{\partial y} \) and \( \frac{\partial H}{\partial y'} \) are to be viewed as the partial derivatives of \( H \) with respect to its second and third arguments, respectively.

Notice that

\[ (2.11) \quad \frac{\Delta H}{\delta y} = \frac{\delta F}{\delta y}, \]

while

\[ (2.12) \quad \frac{\Delta H}{\delta y'} = \frac{\delta F}{\delta y'} - \lambda. \]

Substitution from (2.12) and (2.10) into (2.8) yields:

\[ (2.13) \int_0^{x_2} \frac{\delta F}{\delta y'} (\delta y) + \frac{\delta F}{\delta y'} \frac{d}{dx} (\delta y) - \lambda \frac{d}{dx} (\delta y) dx = 0. \]

Now

\[ (2.14) \int_0^{x_2} \lambda \frac{d}{dx} (\delta y) dx = \lambda [\delta y]_0^{x_2}. \]

It is part of the variational method that \( \delta y \) should vanish identically at any known end point; thus \( \delta y(0) = 0 \). Hence (2.13) becomes:

\[ \sum_{n=1}^{N} n(10) > 0 \]

Extraction Rate

\[ y'(x), K^* \]

\[ y'(x), K^* \]

Figure 3 The Possible Effects of an Expected Sustained Increase in the Mineral Price on the Optimal Extraction Path

The initial expected price path is \( p(x) = K^* e^{rx} \); the shocked path is \( p(x) = K^* e^{rx} \), where \( K^* = 1.01 K_c \).

5. PROPOSED APPLICATION TO ORANI

5.1 Narrowing Down the Range of Relevant Possibilities

The foregoing results are disappointing to the extent that the contemporaneous comparative static price elasticities of supply from a mine at an arbitrary future date cannot even be signed without specific information on parameters. Since the mining sector in Australia is relatively highly concentrated, for reasons of commercial secrecy it is unlikely that engineering and/or cost information which would allow the estimation of mine-specific cost functions will be obtainable.

One approach for further work is as follows. Since the cost parameters \( a, y^*, \delta \) and \( \bar{e} \) are not likely to be estimable, we could concentrate instead on the planned shut-down date \( x_2 \). This is a quantity about which general information may be obtainable. As we have seen above, treating \( K = p(0) \) as arbitrary results in no loss of generality (and in any event, the current price of the mineral is known). The market discount rate \( r \), and the expected rate of price increase for the mineral, \( n \), should be more or less discernable from market data and trade literature. Treating \( K, r, n \) and \( x_2 \) as known, by a process of trial and error we could find an arbitrary cost parameter vector.
\[
\begin{align*}
\frac{\partial^2 y}{\partial x^2} &= (y')' = y'' \\
\frac{\partial^2 y}{\partial x^2} &= y'' = \frac{\partial}{\partial x} \left( \frac{\partial y}{\partial x} \right) = \frac{\partial^2 y}{\partial x^2}
\end{align*}
\]
where $\frac{\partial h}{\partial x} = h'$, it follows from (2.17b) that

$$\frac{\partial y}{\partial x} = h'(x_2, \lambda)$$

(2.20)

$$h'(x_2, \lambda) = \frac{H(x_2, \lambda)}{1-x_2} = 0$$

Since $F$ is not a function of $\lambda$, from (2.7') we see that

$$\frac{\partial y}{\partial \lambda} = [a - y(x_2)]$$

(2.21)

thus (2.17a) yields (2.6) again.

The expression $[\frac{\partial F}{\partial y'} \frac{d}{dx} (\delta y)]$ contained on the right of (2.18) may be integrated by parts to yield:

$$\int \frac{\partial F}{\partial y'} \frac{d}{dx} (\delta y) \, dx = \frac{\partial F}{\partial y'} (\delta y) - \int (\delta y) \frac{d}{dx} \left( \frac{\partial F}{\partial y'} \right) \, dx$$

(2.22)

Since $y(0) = 0$,

$$\left[ \frac{\partial F}{\partial y'} (\delta y) \right]_0^{x_2} = \left[ \frac{\partial F}{\partial y'} \right]_{x=x_2} \delta y(x_2)$$

(2.23)

Using (2.22) and (2.23) in substitutions into (2.17b), we obtain:

$$\delta y = \int_0^{x_2} \left[ \frac{\partial F}{\partial y'} (\delta y) - \frac{d}{dx} \left( \frac{\partial F}{\partial y'} \right) \right] (\delta y) \, dx + \left[ \frac{\partial F}{\partial y'} (\delta y) \right]_{x=x_2} = 0$$

(2.24)

Kuhn-Tucker tells us that exactly one of the following holds:

$$a > y(x_2), \lambda = 0$$

(2.25a)

or

$$\lambda > 0, a = y(x_2)$$

(2.25b)

If (2.25a) holds, (2.24) reduces to

Table 7

| RESPONSE OF SUPPLY IN THE LONG RUN (10 YEARS) TO A SUSTAINED CHANGE |
| IN THE EXPECTED PRICE OF THE MINERAL (a) |

| Search over Parameter Set A |
| Number of possible combinations tried: 729 |
| Number of valid solutions found for $x_2$: 367(b) |
| **Elasticity of $y'(10)$ with Respect to $K$** |
| Minimum Elasticity | -0.0568(c) |
| Maximum Elasticity | 1.4636(d) |
| Average Elasticity | 0.9308 |
| Mean Absolute Deviation | 0.2616 |

| Search over Parameter Set B |
| Number of possible combinations tried: 729 |
| Number of valid solutions found for $x_2$: 729 |
| **Elasticity of $y'(10)$ with Respect to $K$** |
| Minimum Elasticity | 0.0423(e) |
| Maximum Elasticity | 1.5035(f) |
| Average Elasticity | 0.9823 |
| Mean Absolute Deviation | 0.2416 |

(a) This table should be read in conjunction with the multiple parameter sets listed in Table 5.

(b) The remaining combinations gave values for $x_2$ of less than ten years, and have thus been excluded from analysis.

(c), (d), (e), (f): The parameter sets generating these extrema were:

<table>
<thead>
<tr>
<th>$K$</th>
<th>$a$</th>
<th>$y^*$</th>
<th>$\theta$</th>
<th>$\rho$</th>
<th>$n$</th>
<th>$\bar{y}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(c)</td>
<td>104.5</td>
<td>200,000</td>
<td>1.96</td>
<td>20.0</td>
<td>0.01</td>
<td>0.001</td>
</tr>
<tr>
<td>(d)</td>
<td>104.5</td>
<td>50,000</td>
<td>0.010</td>
<td>20.0</td>
<td>0.01</td>
<td>-0.05</td>
</tr>
<tr>
<td>(e)</td>
<td>150.0</td>
<td>200,000</td>
<td>5</td>
<td>20.0</td>
<td>0.01</td>
<td>0.001</td>
</tr>
<tr>
<td>(f)</td>
<td>150.0</td>
<td>50,000</td>
<td>1</td>
<td>20.0</td>
<td>0.01</td>
<td>0.001</td>
</tr>
</tbody>
</table>
\[
\begin{align*}
T_p (\frac{\partial}{\partial x}) &= \frac{T_{n(x)}}{\partial x} \\
\frac{\partial}{\partial x} &= \frac{\partial}{\partial x} \\
\end{align*}
\]
that is, with an obvious change of notation,

$$(2.30) \sum_{i=1}^{n} (A_i + B_i)u_i = 0 .$$

But, since the $u_i$ are free parameters, the only constraint of the kind (2.30) which is permissible is the trivial one in which $A_i + B_i = 0$. That is,

$$(2.31) \left[ A(x) - \lambda \right] \frac{\partial}{\partial u_i} \psi(u, x) = \int_{0}^{x} B(x) \frac{\partial}{\partial u_i} \psi(u, x) dx (i = 1, \ldots, n).$$

Differentiate (2.31) with respect to $x_i$, obtaining:

$$(2.33) \left[ A'(x) + B(x) - \lambda' \right] \frac{\partial}{\partial x_i} \psi(u, x) = \int_{0}^{x} B(x) \frac{\partial}{\partial x_i} \psi(u, x) dx (i = 1, \ldots, n),$$

where $A'(x)$ is the derivative of $A$ with respect to $x_i$. Thus

$$B(x) = \frac{\partial}{\partial x_i} \left[ A(x) - \lambda \right] \frac{\partial}{\partial u_i} \psi(u, x)$$

$$(2.34) \frac{\partial}{\partial x_i} \psi(u, x) = \frac{\partial}{\partial x_i} \left[ A'(x) + B(x) - \lambda' \right] \frac{\partial}{\partial u_i} \psi(u, x) (i = 1, \ldots, n).$$

Substituting from (2.34) into (2.33):

$$(2.35) \frac{\partial}{\partial x_i} \left[ A'(x) + B(x) - \lambda' \right] \frac{\partial}{\partial u_i} \psi(u, x)$$

$$= \left[ A - \frac{\partial}{\partial x_i} \right] \frac{\partial}{\partial u_i} \psi(u, x) (i = 1, \ldots, n).$$

Suppose

$$\frac{\partial}{\partial x_i} \psi(u, x) \uparrow \lambda_i \downarrow x_i, \quad \frac{\partial}{\partial x_i} \psi(u, x) \downarrow \lambda_i \downarrow x_i$$

Then

In Table 6 the elasticities of the shut-down date with respect to the parameters of the problem are reported for a subset of eight parameter combinations. Because of its role in computing $n(t)$ (see (3.34)), the first column of Table 6 is of particular interest.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Set</th>
<th>$K$</th>
<th>$C^*$</th>
<th>$Y^*$</th>
<th>$\theta$</th>
<th>$\alpha$</th>
<th>$n$</th>
<th>$p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_s$ with Respect to:</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$K$</td>
<td>$C^*$</td>
<td>$Y^*$</td>
<td>$\theta$</td>
<td>$\alpha$</td>
<td>$n$</td>
<td>$p$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>-----------</td>
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<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Table 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Set I</td>
</tr>
<tr>
<td>Set II</td>
</tr>
<tr>
<td>Set III</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Table 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Set A1</td>
</tr>
<tr>
<td>Set AII</td>
</tr>
<tr>
<td>Set AIII(b)</td>
</tr>
</tbody>
</table>

(b) Not computed.

4.4 Comparative Static Long-Run Price Elasticities of Supply

All possible parameter combinations selected from the multiple sets listed in Table 5 have been tested for valid solutions to $x_s$, and the elasticity $n(t)$ of the rate of extraction in year 10, $Y'(10)$, with respect to $K$ has been computed. Table 7 records an analysis of these elasticities for the 367 and 729 valid solutions found for parameter.
The parameter vector immediately above the column \( x^2 \) is the solution to the equation (6) of \( \tau \times T_x \) and corresponds to the equation for the determinant of the matrix.

The initial slope is used as a measure and has this relationship, from parameters set \( A \) and parameters set \( B \), the combination of the parameters selected.

The result is shown in each case over the error at \( 1.0 \).
(2.41a) Average PV Profit at end-point = Marginal PV Profit at end-point

i.e., when \( y'(x) = 0 \),

(2.41b) \( F(x)/y'(x) = \frac{dF}{dy} x \)

thus in the case of surplus reserves, average PV profit is maximized at the terminal point. When \( y'(x) \) is zero, (2.41b) continues to hold in a limiting sense as \( y'(x) \to 0 \) (see Levhari and Livinston (1977, pp. 180 and 181)). If there is no shortage of reserves, and (2.25a) applies, the marginal PV profit at the terminal point, \( [dF/dy]'_x = D_x \), is driven to zero, so that \( F(x) = 0 \). These results, together with results for the case in which \( x \) is known, are summarized in Table 2.

2.4 Interpretation of Euler Equation

We conclude this section with a brief discussion of Levhari and Liviston's (1977) interpretation of the Euler Equation (2.39), which may be rewritten:

(2.42) \( e^{-rx} \frac{dH}{dy} = \frac{dx}{dX} \left[ e^{-rx} \frac{dH}{dy} \right] \).

Integrate (2.42) over the interval \( [x, x'] \), where \( x \) is arbitrary (but within \( [0, x] \)), and multiply by \( e^{rx} \):

(2.43) \[ x e^{-r(t-x)} \frac{dH}{dy} \left. \frac{dt}{t} \right|_t = \int_x^{x'} \left[ e^{-r(t-x)} \frac{dH}{dy} \right. \left. \frac{dt}{t} \right] = \left[ e^{-r(t-x)} \frac{dH}{dy} \right]_t^{x'} \],

Table 4

THREE NUMERICAL SOLUTIONS FOR THE SHUT-DOWN DATE IN THE CASE OF A QUADRATIC/LINEAR COST FUNCTION

<table>
<thead>
<tr>
<th>Parameters</th>
<th>I</th>
<th>II</th>
<th>III</th>
</tr>
</thead>
<tbody>
<tr>
<td>( K )</td>
<td>$104.50/ton</td>
<td>$104.50/ton</td>
<td>$104.50/ton</td>
</tr>
<tr>
<td>( a )</td>
<td>$200,000.00/year</td>
<td>$200,000.00/year</td>
<td>$200,000.00/year</td>
</tr>
<tr>
<td>( y' )</td>
<td>$1.96/ton</td>
<td>$1.96/ton</td>
<td>$1.96/ton</td>
</tr>
<tr>
<td>( \theta )</td>
<td>$10.0/ton</td>
<td>$10.0/ton</td>
<td>$10.0/ton</td>
</tr>
<tr>
<td>( r )</td>
<td>0.07% per year</td>
<td>0.07% per year</td>
<td>0.07% per year</td>
</tr>
<tr>
<td>( n )</td>
<td>-0.05% per year</td>
<td>-0.02% per year</td>
<td>0.00001% per year</td>
</tr>
<tr>
<td>( \beta )</td>
<td>$0.00001/ton</td>
<td>$0.00001/ton</td>
<td>$0.00001/ton</td>
</tr>
</tbody>
</table>

Solution

\( x \) = 10.7 years, 16.1 years, 20 years

Notes: (a) The optimal extraction paths correspond to Panels I, II and III (respectively) of Figure 1.
The marginal profit at the point of the optimal solution is zero. The coordinates of the optimal solution are (x, y). The profit at this point is $P(x, y)$.

In the optimal solution, the profit function takes the form $P(x, y) = ax^2 + by^2 + cxy + dx + ey + f$. The coefficients a, b, c, d, e, and f are determined by the constraints of the problem.

### Table 2

<table>
<thead>
<tr>
<th>x</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>y</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
</tbody>
</table>

### Scenario 1

- **Scenario 1**: When the constraints are active.
  - Objective function: $P(x, y) = ax^2 + by^2 + cxy + dx + ey + f$
  - Constraints: $g_1(x, y) = 0$, $g_2(x, y) = 0$
  - The optimal solution is at the intersection of the constraints.

### Scenario 2

- **Scenario 2**: When the constraints are inactive.
  - Objective function: $P(x, y) = ax^2 + by^2 + cxy + dx + ey + f$
  - Constraints: $g_1(x, y) = 0$, $g_2(x, y) = 0$
  - The optimal solution is a point where the profit function is maximized subject to the constraints.
where \( \Pi(\cdot) \) indicates the marginal profitability of extraction at \( \cdot \). Consider now the left of (2.43). Since there is no reason to suppose that revenue, \( R \), depends on cumulative extractions \( y \), \( 2\Pi/\Pi y \) may be replaced with \( -2C/3y \), where \( C \) is total cost. Thus the left of (2.43) may be interpreted as minus the cumulated cost disability from \( x \) to the end of the plan of an additional unit of (cumulative) extraction at \( x \). In Hotelling (1931), \( 2C/3y = 0 \), and (2.44) leads to the famous result that on an optimal plan, marginal profit must grow at the rate of discount. The first right hand term of (2.44) can be written:

\[
(2.45) \quad e^{-\tau(x-x)} \Pi(x) = PV_x \{ \Pi(x) \} ;
\]

that is, "the present value at \( x \) of the marginal profit at the end of the plan". Rearranging (2.44) we have:

\[
(2.46) \quad \Pi(x) = PV_x \{ \Pi(x) \} + PV_x \{ CCD(x,x_2) \} ,
\]

where the last term of (2.46) is just the negative of the left-hand side of (2.44), and is to be interpreted as "the present value at \( x \) of the cumulated cost disability over the remainder of the plan caused by the extraction of an additional unit at \( x \)". Splitting marginal profit into revenue \( MR \) and marginal cost \( MC \), (2.46) may be rewritten:

\[
(2.47) \quad MR(x) = MC(x) + PV_x \{ \Pi(x) \} + PV_x \{ CCD(x,x_2) \} = FMC(x) \text{ (say)},
\]

where \( FMC \) is "full marginal cost", which is the sum of three components:

(a) ordinary short-run marginal cost at a fixed level of cumulative extractions;

(b) the opportunity cost of not waiting till the end of the plan to mine the last unit extracted in the current period;

(c) the future cost penalties incurred by mining the last unit in the current period.

4.3 Establishing a Parameter Set

In order to evaluate the quadratic/linear cost function and to solve for \( x_2 \), we must have information on the relevant interest rate for discounting, \( r \); the cost function parameters \( a \), \( b \), \( c \), \( \alpha \), and \( \gamma \); and finally on the initial product price \( K \) and on its expected rate of growth, \( n \). The term \( a \) is fixed costs, the terms \( b \) and \( c \) relate to current extraction costs at any given level of cumulative extractions, whilst \( \gamma \) represents the upward shift in the cost function for each additional unit of cumulative extractions (see (3.3)).

The selection of the nominal growth rates \( n \) and \( r \) imply a time unit of measurement for \( x \). We evaluate \( n \) and \( r \) such that \( x \) is measured in years. We are not interested in the case of \( n = r \) as this implies that there is no profitable use of a resource is to leave it unmined, appreciating in value. The value of \( K \) (i.e., the initial price \( p_0 \)) is used as a numerator: solutions depend only on costs relative to price. Thus solutions are homogeneous of degree zero in \((a/K, b/K, c/K, \gamma/K)\).

Whilst, hopefully, our quadratic/linear cost function has application to Australian extractive industries, it is not our objective here to attempt any econometric estimations using empirical data. Rather, we seek first to establish how extensive the range of parameter values are which will provide valid solutions in the computation of \( x_2 \).

The criteria for validity of a solution are as follows:

1. Output \( y(x) \) must be non-negative at all instants \( x \) in the plan \( (x \in [0, x_2]) \). (This is not guaranteed using the classical methods described above in Section 3.)

2. In addition to the first-order conditions identified in Table 2, appropriate second-order conditions must be satisfied (since the first-order conditions alone sometimes identify a saddle point; i.e., a maximum with respect to the form of \( y(x) \) but a minimum with respect to the shut-down date \( x_2 \); see the Appendix).
Order of this section is a function of the model described.

Relation of Models and the Property of the Model are Functions of the Model and are Functions of the Properties of the Model.

We have already noted that (9) stands for a line with a slope of one. In this case when

\[ y = \frac{1}{x} \]
3.2 Specific Assumptions

We continue with the convention $x_1 = 0$. The specific assumptions to be made in what follows are:

$$p(x) = Ke^{nx},$$

(3.1)

(the price $p$ of the mineral is expected to grow at the uniform rate $n$, starting from an initial value of $p(0) = K$);

$$\Pi(x, y, y') = p(x) y'(x) - C(y'(x), y(x)).$$

(3.2)

where

$$C(y'(x), y(x)) = a + \alpha y'(x) + \frac{1}{2} \beta [y'(x)]^2 + \gamma y(x),$$

$$\alpha, \beta, \gamma, a \geq 0.$$ (3.3)

In (3.2), $p(x) y'(x)$ is gross revenue flow at time $x$, and $C$ is cost flow at that time. Equation (3.3) makes it clear that there are fixed costs $a$; and that when the rate of extraction $y'$ increases, costs rise, with marginal costs varying linearly with $y'$ at fixed $y$. Also, the rate of total cost flow rises linearly with the extent $y$ of existing depletion ($y' > 0$).

3.3 A Particular Solution

The right-hand side of the Euler Condition (2.39) in the present case is:

$$\frac{d}{dx} \left( \frac{3F}{3F/3F} \right) = \frac{d}{dx} \left[ e^{-rx} \frac{3H}{3F/3F} \right]$$

$$= \frac{d}{dx} \left[ e^{-rx} \left( p(x) - \frac{3C}{3F/3F} \right) \right]$$

$$= \frac{d}{dx} \left[ e^{-rx} \left( p(x) - \delta y'(x) - \theta \right) \right]$$

$$= e^{-rx} \left[ p'(x) - \delta y''(x) \right] - re^{-rx} \left( p(x) - \delta y'(x) - \theta \right)$$

The optimal extraction path implied by (4.1) may be I: decline, II: decline then growth, or III: growth. It is not possible, given our quadratic-linear cost function, to have IV: a concave optimal extraction path. Figure 1 demonstrates these characteristics of equation (4.1).

```
<table>
<thead>
<tr>
<th></th>
<th>I</th>
<th>II</th>
<th>III</th>
<th>IV*</th>
</tr>
</thead>
<tbody>
<tr>
<td>qty per unit time</td>
<td>y'(x)</td>
<td>y'(x)</td>
<td>y'(x)</td>
<td>y'(x)</td>
</tr>
<tr>
<td>0</td>
<td>x</td>
<td>0</td>
<td>x</td>
<td>x</td>
</tr>
</tbody>
</table>
```

Figure 1: Optimal Extraction Paths: Three Solution Paths and One Non-Solution Path. I, II and III are solution paths to equation (3.3). IV* is not a solution.

The first term on the right of equation (4.1), which is unambiguously positive, is composed of a growth variable $e^{rx}(x-x_1)$, and a coefficient $y'(r)(r^3)$. The entire term reaches a maximum value of $y'(r)(r^3)$ when $x = x_2$. One possible path for the first term in equation (4.1) is plotted in Panel I of Figure 2. The second term, $e^{rx}(r^3)$, also is ambiguously positive, but it may either grow ($n > 0$) or decline ($n < 0$). The initial value of the term (at $x = 0$) is $K/8$. When $n$ is negative the declining growth path demonstrated in Panel I of Figure 1 is possible. However, the positive growth effects of $r$ may eventually overtake that of negative growth (due to $n < 0$), with the optimal extraction path declining and then growing (see Panel II, Figure 1). The positive growth due to the first term may dominate when $n < 0$, even from time $x = 0$. In this latter case, Panel III of Figure 1 would be the outcome. Finally, it is noted that as all of the elements of the third term in equation (4.1) are positive, the entire term remains negative.
where

\[
\frac{\partial^2}{\partial x^2} y = (x) \quad (3.9)
\]

Substituting from (3.7) into (3.7), and rearranging, we obtain:

\[
\{y(x) - (x)\} \frac{\partial^2}{\partial x^2} y = (x) \quad (3.10)
\]

By the definition of (3.7), we have

\[
p(x) \cdot \{x - (x)\} \frac{\partial}{\partial x} y = (x) \quad (3.11)
\]

This equation is obtained from (3.7) by substituting (3.10) for (3.7). The procedure of extraction presented in this paper is a formal procedure for solving (3.10).

From (3.11) we see that if (3.10) we obtain

\[
\frac{\partial}{\partial x} y = (x) \quad (3.12)
\]

where

\[
\frac{\partial}{\partial x} y = (x) \quad (3.13)
\]

Substituting from (3.12) into (3.10), we obtain an ordinary differential equation (3.12)

\[
\frac{\partial}{\partial x} y = (x) \quad (3.14)
\]

The left-hand side of the ordinary differential equation (3.12) is:

\[
\frac{\partial}{\partial x} y = (x) \quad (3.15)
\]

(3.15)

\[
\frac{\partial}{\partial x} y = (x) \quad (3.16)
\]

The right-hand side of the ordinary differential equation (3.12) is:

\[
\frac{\partial}{\partial x} y = (x) \quad (3.17)
\]

(3.17)

\[
\frac{\partial}{\partial x} y = (x) \quad (3.18)
\]

(3.18)

\[
\frac{\partial}{\partial x} y = (x) \quad (3.19)
\]

(3.19)

\[
\frac{\partial}{\partial x} y = (x) \quad (3.20)
\]

(3.20)

\[
\frac{\partial}{\partial x} y = (x) \quad (3.21)
\]

(3.21)

\[
\frac{\partial}{\partial x} y = (x) \quad (3.22)
\]

(3.22)

\[
\frac{\partial}{\partial x} y = (x) \quad (3.23)
\]

(3.23)

\[
\frac{\partial}{\partial x} y = (x) \quad (3.24)
\]

(3.24)

\[
\frac{\partial}{\partial x} y = (x) \quad (3.25)
\]

(3.25)

\[
\frac{\partial}{\partial x} y = (x) \quad (3.26)
\]

(3.26)

\[
\frac{\partial}{\partial x} y = (x) \quad (3.27)
\]

(3.27)

\[
\frac{\partial}{\partial x} y = (x) \quad (3.28)
\]

(3.28)

\[
\frac{\partial}{\partial x} y = (x) \quad (3.29)
\]

(3.29)

\[
\frac{\partial}{\partial x} y = (x) \quad (3.30)
\]

(3.30)

\[
\frac{\partial}{\partial x} y = (x) \quad (3.31)
\]

(3.31)

\[
\frac{\partial}{\partial x} y = (x) \quad (3.32)
\]

(3.32)

\[
\frac{\partial}{\partial x} y = (x) \quad (3.33)
\]

(3.33)

\[
\frac{\partial}{\partial x} y = (x) \quad (3.34)
\]

(3.34)

\[
\frac{\partial}{\partial x} y = (x) \quad (3.35)
\]

(3.35)

\[
\frac{\partial}{\partial x} y = (x) \quad (3.36)
\]

(3.36)

\[
\frac{\partial}{\partial x} y = (x) \quad (3.37)
\]

(3.37)

\[
\frac{\partial}{\partial x} y = (x) \quad (3.38)
\]

(3.38)

\[
\frac{\partial}{\partial x} y = (x) \quad (3.39)
\]

(3.39)

\[
\frac{\partial}{\partial x} y = (x) \quad (3.40)
\]

(3.40)

\[
\frac{\partial}{\partial x} y = (x) \quad (3.41)
\]

(3.41)

\[
\frac{\partial}{\partial x} y = (x) \quad (3.42)
\]

(3.42)

\[
\frac{\partial}{\partial x} y = (x) \quad (3.43)
\]

(3.43)

\[
\frac{\partial}{\partial x} y = (x) \quad (3.44)
\]

(3.44)

\[
\frac{\partial}{\partial x} y = (x) \quad (3.45)
\]

(3.45)

\[
\frac{\partial}{\partial x} y = (x) \quad (3.46)
\]

(3.46)

\[
\frac{\partial}{\partial x} y = (x) \quad (3.47)
\]

(3.47)

\[
\frac{\partial}{\partial x} y = (x) \quad (3.48)
\]

(3.48)

\[
\frac{\partial}{\partial x} y = (x) \quad (3.49)
\]

(3.49)

\[
\frac{\partial}{\partial x} y = (x) \quad (3.50)
\]

(3.50)

\[
\frac{\partial}{\partial x} y = (x) \quad (3.51)
\]

(3.51)

\[
\frac{\partial}{\partial x} y = (x) \quad (3.52)
\]

(3.52)

\[
\frac{\partial}{\partial x} y = (x) \quad (3.53)
\]

(3.53)
\[ (3.9') \quad Y = Y^* + r_0. \]

This says \( Y = (Y^*/g) \) grows exponentially at the rate \( n \). A particular solution of (3.9) is found by noting that (3.9) is compatible with \( (y' + \frac{Y}{B}) \) growing exponentially at the rate \( n \). Proof: Let

\[ (3.10) \quad y'(x) + \frac{Y}{B} = Ae^{nx}. \]

Then

\[ (3.11) \quad y(x) = n Ae^{nx}. \]

Substitute from (3.10) and (3.11) into (3.5b):

\[ (3.12) \quad Y(x) = n Ae^{nx} - r[Ae^{nx} - Y/(B)]; \]

that is,

\[ (3.13) \quad y''(x) - r y'(x) = \frac{Y}{B} + (n - r) Ae^{nx}. \]

Comparing (3.9) and (3.13), we see that \( A = K/B \). Integrating (3.10) with \( A = K/B \), we see that:

\[ (3.14) \quad y(x) = \frac{K e^{nx}}{(nB)} - \frac{Yx}{(B)} + \text{const}, \]

is a particular solution for \( y \).

3.4 General Solution

The second-order linear differential equation formed from the homogeneous part of (3.13), whose characteristic polynomial (a quadratic) is:

\[ (3.15) \quad 1z^2 - rz = 0, \]

has roots \( r \) and zero. Accordingly, the general solution for the homogeneous part of (3.13) is (see, e.g., Cogan and Norman (1958, p. 207)):

\[ (3.16) \quad y(x) = \text{const}_1 e^{rx} + \text{const}_2. \]

Adding the particular solution (3.14) to the general solution (3.16) of the homogeneous part of (3.13), we find (3.13)'s general solution to be:

\[ (3.32) \quad \frac{3\text{ln} y'(t)}{3\text{ln} p(t)} = \frac{n(t)}{\text{ln} K} \quad (\text{all } t). \]

Differentiating (3.27) with respect to \( x \), we obtain

\[ (3.33) \quad y'(x) = e^{r(x-x_2)} \frac{y}{(rB)} + \frac{K}{B} e^{nx} - \frac{Y}{(rB)}. \]

With \( x \) fixed \( (x \in [0, x_2]) \), differentiate (3.33) with respect to \( K \) (whilst keeping in mind that \( x_2 \) is a function of \( K \)):

\[ (3.34) \quad \frac{3y'(x)}{3K} = -\frac{y}{B} e^{r(x-x_2)} \frac{3x_2}{3K} + \frac{enx}{B}. \]

We now develop an expression for \( 3x_2/3K \) from the listing in Table 3 of the coefficients of variables involving \( x_2 \) in the equation obtained by equating (3.28) and (3.29). We see that

\[ (3.35) \quad A \frac{3}{3K} \left[ e^{2nx} + 2nx \frac{3A}{3K} + B \frac{3}{3K} \left( e^{nx} + \frac{3B}{3K} \right) \right] + C \frac{3}{3K} \left( e^{-rX} + e^{-rx} \frac{3C}{3K} + D \frac{3x}{3K} + \frac{3E}{3K} \right) = 0. \]

From Table 2 we see that \( 3C/3K = 0 \), while

\[ (3.36a) \quad \frac{3A}{3K} = \frac{K}{(gY^*)}, \]

\[ (3.36b) \quad \frac{3B}{3K} = -\frac{1}{B} \left( \frac{B}{Y^*} + \frac{1}{n} \right), \]

and

\[ (3.36c) \quad \frac{3E}{3K} = 1/(nB). \]

If follows that

\[ (3.37a) \quad 3x_2/3K = \left[ \frac{1}{n} + \frac{Y}{K} \right] / \left[ (1 - \frac{Y}{r}) e^{-rX} + \frac{Y}{r} \right], \]

where
but from (3.17)

\[ \frac{g}{x} = \frac{f(x)}{x} \]  

(3.22)

dо

\[ 0 = \left[ \left( f(x), A \right) \right] d + \left( x, A \right) d + \left( x, A \right) d = \left( x, A \right) d = \frac{\sigma}{\lambda} \]  

(3.17)

That is,

\[ \left( f(x), A \right) \frac{\sigma}{\lambda} = \left( x, A \right) \frac{\sigma}{\lambda} = \left( x, A \right) \frac{\sigma}{\lambda} \]

Working time with (3.19), we have

\[ 0 = \frac{\sigma}{\lambda} \]  

(3.3)

\[ 0 = \left[ \left( f(x), A \right) \right] d + \left( x, A \right) d + \left( x, A \right) d = \left( x, A \right) d = \frac{\sigma}{\lambda} \]  

(3.19)

3.6 The Elasticity of Mining Output with Respect to a Downward Price

\[ \frac{\partial y}{\partial x} = \frac{x(x,y) - xy}{x} = \frac{xy + x}{y} \]

(3.20)

\[ \frac{\partial y}{\partial x} = \frac{x}{y} \]

(3.21)

\[ \frac{\partial y}{\partial x} = \frac{x}{y} \]

(3.22)

\[ \frac{\partial y}{\partial x} = \frac{x}{y} \]

(3.23)

\[ \frac{\partial y}{\partial x} = \frac{x}{y} \]

(3.24)

\[ \frac{\partial y}{\partial x} = \frac{x}{y} \]

(3.25)

\[ \frac{\partial y}{\partial x} = \frac{x}{y} \]

(3.26)

\[ \frac{\partial y}{\partial x} = \frac{x}{y} \]

(3.27)

\[ \frac{\partial y}{\partial x} = \frac{x}{y} \]

(3.28)

\[ \frac{\partial y}{\partial x} = \frac{x}{y} \]

(3.29)

\[ \frac{\partial y}{\partial x} = \frac{x}{y} \]

(3.30)

\[ \frac{\partial y}{\partial x} = \frac{x}{y} \]

(3.31)

\[ \frac{\partial y}{\partial x} = \frac{x}{y} \]

(3.32)

\[ \frac{\partial y}{\partial x} = \frac{x}{y} \]

(3.33)

\[ \frac{\partial y}{\partial x} = \frac{x}{y} \]

(3.34)

\[ \frac{\partial y}{\partial x} = \frac{x}{y} \]

(3.35)

\[ \frac{\partial y}{\partial x} = \frac{x}{y} \]

(3.36)

\[ \frac{\partial y}{\partial x} = \frac{x}{y} \]

(3.37)

\[ \frac{\partial y}{\partial x} = \frac{x}{y} \]

(3.38)

\[ \frac{\partial y}{\partial x} = \frac{x}{y} \]

(3.39)

\[ \frac{\partial y}{\partial x} = \frac{x}{y} \]

(3.40)

\[ \frac{\partial y}{\partial x} = \frac{x}{y} \]

(3.41)

\[ \frac{\partial y}{\partial x} = \frac{x}{y} \]
(3.23) \[ y'(x) = r \cdot C_1 e^{nx} + \frac{K}{\beta} e^{nx} - \frac{Y}{r} \cdot x. \]

Equating (3.22) and (3.23) we see that

(3.24) \[ C_1 = e^{-rx} \cdot \left( \frac{Y}{r} \right) \]

Using the initial condition \( y(0) = 0 \), from (3.17b) we see that

(3.25) \[ -(C_1 + C_2) = \frac{K}{\beta} \]

thus from (3.24),

(3.26) \[ C_2 = \left( \frac{K}{\beta} \right) e^{rx} \cdot \left( \frac{Y}{r} \right) \]

Substituting from (3.24) and (3.26) into (3.17b), the solution for cumulative extractions \( y(x) \) is:

(3.27) \[ y(x) = e^{r(x-1)} \cdot \left( \frac{Y}{r} \right) + \frac{K}{\beta} e^{nx} - \frac{Y}{r} - \left( \frac{K}{\beta} \right) e^{-rx} \cdot \left( \frac{Y}{r} \right) \]

3.5 The Optimal Planning Horizon

To this point, with the exception of determining the value of \( x \), we have satisfied objective (a) of this section under assumptions (3.1) and (3.3). We now derive an equation which implicitly determines \( x \). This also serves as a preliminary to pursuing objective (b).

If we put \( x = x \) in (3.27) we obtain:

(3.28) \[ y(x) = \frac{1}{\beta} \left[ \frac{K}{\beta} e^{nx} - 1 \right] - \frac{Y}{r} \cdot e^{-rx} - 1 - \frac{Y}{r} \cdot x. \]

Now from (3.20),

\[ y(x) = \frac{1}{\beta} \left[ Ke^{nx} \cdot y'(x) - 0y'(x) - a - \frac{Y}{r} \cdot y'(x) \right] \]

(by substitution from (3.22))

(3.29a) \[ = \frac{1}{\beta} \left[ \frac{K}{\beta} \cdot [y'(x)]^2 - a \right] \]

(3.29b) \[ = \frac{1}{\beta} \left[ Ke^{nx} \cdot [y'(x)]^2 - a \right] \]

Equating (3.28) with (3.29b) we obtain an expression which, in principle, determines \( x \). It is clear that such an expression involves the first, second, and \((-r/n)\)th powers of \( e^{nx} \), as well as \( x \). We cannot, therefore, obtain a closed-form solution for \( x \). For future use, we note that the equation obtained from (3.28) and (3.29), written with zero on the right, has coefficients as set out in Table 3. The roots of this equation should be relatively easily found with the aid of a non-linear equation package.

An alternative is to construct, and then refine, an approximate solution obtained by taking Taylor expansions of \( e^{nx} \), \( e^{2nx} \), and \( e^{-rx} \), about a guess \( X \) of \( x \), while dropping terms of third and higher order. The relevant approximations are:

(3.30a) \[ e^{nx} \approx e^{nx} \cdot e^{n(x-x_0)} = e^{nx} \cdot [1 + n(x-x_0) + \frac{n^2}{2} (x-x_0)^2] \]

(3.30b) \[ e^{2nx} \approx e^{2n} \cdot e^{2n(x-x_0)} = e^{2nx} \cdot [1 + 2n(x-x_0) + 2n^2(x-x_0)^2] \]

and

(3.30c) \[ e^{-rx} \approx e^{-rx} \cdot e^{-r(x-x_0)} = e^{-rx} \cdot [1 - r(x-x_0) + \frac{r^2}{2} (x-x_0)^2] \]

Use of these equations, and Table 3, leads to a quadratic in \( x \).